A PARTIAL ORDERING OF WEAK POSITIVE QUADRANT DEPENDENCE

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Abstract. A partial ordering is developed among weakly positive quadrant dependent (WPQD) bivariate random vectors. This permits us to measure the degree of WPQD-ness and to compare pairs of WPQD random vectors. Some properties and closures under certain statistical operations are derived. An application is made to measures of dependence such as Kendall's \( \tau \) and Spearman's \( \rho \).

1. Introduction

A bivariate random vector \((X,Y)\) or its distribution \(H\) is positive quadrant dependent (PQD) if \(P[X > x, Y > y] - P(X > x)P(Y > y) \geq 0\) for all \(x,y\). Lehmann (1966) introduced this concept of positive quadrant dependence (PQD) together with some other dependence concepts. For review of some dependence concepts, one may consult Barlow and Proschan (1981) or Tong (1980). For recent literature one may also consult Jogdeo (1982).

Dependence concepts introduced in the literature are mostly stronger than positive quadrant dependence. Recently, Alzaid (1990) introduced a notion of weak positive quadrant dependence (WPQD) between two random variables. The importance of this concept of dependence lies in the fact that it is weaker than the positive quadrant dependence and it enjoys various properties. For many purposes, in addition to knowledge of the nature of dependence, it is also important to know the degree of WPQD-ness and to compare pairs of WPQD random vectors as to their WPQD-ness. Ahmed et al. (1979) have studied very extensively...
the partial ordering of positive quadrant dependence which permits us to compare pairs of positive quadrant dependent bivariate random vectors with specified marginals as to their PQD-ness. Kimeldorf, G. and Sampson, A.R(1987) presented a systematic basis for studying orderings of bivariate distributions according to their degree of positive dependence and introduced a general concept of a positive dependence ordering.

In this paper a partial ordering of weak positive quadrant dependence is developed to compare pairs of weakly positive quadrant dependent bivariate random vectors. The definitions and some basic properties of WPQD ordering are introduced in Section 2. Some preservation results of WPQD ordering are derived in Section 3. It is shown that the WPQD ordering is preserved under convolution, mixture of a certain type, transformation, and limit in distribution. We also make an application of WPQD ordering to some well known quantitative measures of dependence such as Kendall’s $\tau$ and Spearman’s $\rho$.

2. Preliminaries

In this section we present definitions, and basic facts used in the sequel.

**Definition 2.1.** (Alzaid, 1990) A bivariate random vector $(X, Y)$ or its distribution function $H$ is said to be weakly positive quadrant dependent of the first type(WPQD1) (second type (WPQD2)) if for all $x, y$.

\begin{align}
& (2.1) \quad \int_x^\infty \int_y^\infty [P(X > u, Y > t) - P(X > u)P(Y > t)]dtdu \geq 0. \\
& \quad (2.1)' \quad \left( \int_{-\infty}^x \int_{-\infty}^y [P(X > u, Y > t) - P(X > u)P(Y > t)]dtdu \geq 0 \right)
\end{align}

A bivariate random vector $(X, Y)$ or its distribution $H$ is said to be weakly positive quadrant dependent(WPQD) if it is WPQD1 and WPQD 2. Let $\beta = \beta(F,G)$ denote the class of bivariate distribution functions $H$ on $\mathbb{R}^2$ having marginals $F$ and $G, H(x,y) = P(X > x, Y > y)$, and $\beta$ be the subclass of $\beta$ where $H$ is WPQD.
Definition 2.2. Let $H_1$ and $H_2$ both belong to $\tilde{\beta}$. The bivariate random vector $(X_1, X_2)$ or its distribution $H_1$ is said to be more weakly positive quadrant dependent of the first type (second type) than the bivariate random vector $(Y_1, Y_2)$ or its distribution $H_2$ if for all $x, y$,

(2.2) \[ \int_x^\infty \int_y^\infty \bar{H}_1(u, t)dtdu \geq \int_x^\infty \int_y^\infty \bar{H}_2(u, t)dtdu. \]

(2.2)' \[ \left( \int_{-\infty}^x \int_{-\infty}^y \bar{H}_1(u, t)dtdu \geq \int_{-\infty}^x \int_{-\infty}^y \bar{H}_2(u, t)dtdu \right. \]

where the integrals exist.

We write $H_1 \geq WPQD1H_2$ or $(X_1, X_2) \geq WPQD1(Y_1, Y_2)(H_1 \geq WPQD2 H_2$ or $(X_1, X_2) \geq WPQD2(Y_1, Y_2))$. We say that the bivariate random vector $(X_1, X_2)$ or its distribution $H_1$ is more weakly positive quadrant dependent than $(Y_1, Y_2)$ or its distribution $H_2$ if $H_1$ is more WPQD1 than $H_2$ and more WPQD2 than $H_2$. We write $H_1 \geq WPQDH_2$ or $(X_1, X_2) \geq WPQD(Y_1, Y_2)$.

Remark. An equivalent form of (2.2) ((2.2)') is

(2.3) \[ \int_x^\infty \int_y^\infty H_1(u, t)dtdu \geq \int_x^\infty \int_y^\infty H_2(u, t)dtdu. \]

(2.3)' \[ \left( \int_{-\infty}^x \int_{-\infty}^y H_1(u, t)dtdu \geq \int_{-\infty}^x \int_{-\infty}^y H_2(u, t)dtdu \right. \]

where the integrals exist.

Proof. Since $\bar{H}_i(s, t) = 1 - F(s) - G(t) + H_i(s, t)$ for $i = 1, 2$, $F = 1 - F$ and $G = 1 - G\bar{H}_i(u, t) - \bar{F}(u)G(t) = \bar{H}_i(u, t) - F(u)G(t)$ for $i = 1, 2$. Hence

\[
\begin{align*}
\int_x^\infty \int_y^\infty \bar{H}_1(u, t)dtdu & \geq \int_x^\infty \int_y^\infty \bar{H}_2(u, t)dtdu \\
\Rightarrow \int_x^\infty \int_y^\infty [\bar{H}_1(u, t) - \bar{F}(u)G(t)]dtdu & \geq \int_x^\infty \int_y^\infty [\bar{H}_2(u, t) - \bar{F}(u)G(t)]dtdu
\end{align*}
\]
\[
\Rightarrow \int_x^\infty \int_y^\infty [H_1(u, t) - F(u)G(t)]dtdu \\
\geq \int_x^\infty \int_y^\infty [H_2(u, t) - F(u)G(t)]dtdu \\
\Rightarrow \int_x^\infty \int_y^\infty H_1(u, t)dtdu \geq \int_x^\infty \int_y^\infty H_2(u, t)dtdu
\]

**PROPERTY P1.** Let \( H_1 \) and \( H_2 \) belong to \( \bar{\beta} \). Assume that \( H_1 \) is more positively quadrant dependent than \( H_2 \). Then \( H_1 \) is more weakly positive quadrant dependent than \( H_2 \).

**PROOF.** From the definition of positive quadrant dependent ordering (See Ahmed et al. (1979)) and Definition 2.2 the result follows.

From Definition 2.2 we have the following property:

**PROPERTY P2.** Let \( H_1, H_2, \) an \( H_3 \) belong to \( \bar{\beta} \). Then \( H_1 \geq WPQD H_2 \) and \( H_2 \geq WPQD H_3 \) imply \( H_1 \geq WPQD H_3 \).

**PROPERTY P3.** Let \((X, Y)\) and \((U, V)\) have distributions \(H_1\) and \(H_2\), respectively where \(H_1\) and \(H_2\) belong to \(\bar{\beta}\). Assume \((X, Y)\) is more WPQD than \((U, V)\). Then \((Y, X)\) is more WPQD than \((V, U)\).

**PROOF.** First, note that both \((Y, X)\) and \((V, U)\) have marginals \(G\) and \(F\).

\[
\int_y^\infty \int_x^\infty P(Y \leq u, X \leq t)dtdu = \int_x^\infty \int_y^\infty P(X \leq t, Y \leq u)dudt \\
\geq \int_x^\infty \int_y^\infty P(U \leq t, V \leq u)dudt = \int_y^\infty \int_x^\infty P(V \leq u, U \leq t)dtdu.
\]

Hence \((Y, X) \geq WPQD_1(V, U)\). Similarly, \((Y, X) \geq WPQD_2(V, U)\) and the proof is complete.

**PROPERTY P4.** Let \( H_1, H_2 \) belong to \( \bar{\beta} \). Assume that \( H_1 \) is more WPQD than \( H_2 \). Then, for \(0 \leq a \leq 1\),

\[
H_1 \geq WPQD a H_1 + (1 - a) H_2 \geq WPQD H_2.
\]
PROOF. For $a = 0, 1$ it is clear that (2.4) holds. For $0 < a < 1$,

$$
\lim_{x \to -\infty} \left\{ aH_1(x, y) + (1 - a)H_2(x, y) \right\} = G(y),
$$

$$
\lim_{y \to -\infty} \left\{ aH_1(x, y) + (1 - a)H_2(x, y) \right\} = F(x).
$$

Since $aH_1 + (1 - a)H_2$ is clearly WPQD, it belongs to $\bar{\beta}$. Next note that

$$
\int_x^\infty \int_y^\infty H_1(s, t)dt \, ds = \int_x^\infty \int_y^\infty [aH_1(s, t) + (1 - a)H_1(s, t)]dt \, ds
$$

$$
\geq \int_x^\infty \int_y^\infty [aH_2(s, t) + (1 - a)H_2(s, t)]dt \, ds
$$

$$
= \int_x^\infty \int_t^\infty H_2(s, t)dt \, ds.
$$

Thus $H_1 \geq WPQD_1 aH_1 + (1 - a)H_2 \geq WPQD_1 H_2$, for $0 \leq a \leq 1$. Similarly, $H_1 \geq WPQD_2 aH_1 + (1 - a)H_2 \geq WPQD_2 H_2$, for $0 \leq a \leq 1$. Thus the proof is complete.

**Definition 2.3.** A family of WPQD distributions \( \{H_\lambda(x, y)|\lambda \in \Lambda \subset R\} \) is said to be increasingly WPQD in \( \lambda \) if \( \lambda' > \lambda \) implies \( H_{\lambda'} \geq WPQD H_\lambda \).

**Example 2.4.** A bivariate family of \( H_\lambda(x, y), 0 < \lambda < 1 \) is increasingly WPQD in \( \lambda \), where \( H_\lambda(x, y) = \lambda H(x, y) + (1 - \lambda)FG \) and \( H \in \bar{\beta} \). It is clear that \( H_\lambda \subset \beta \) by Property \( P4 \). For $0 < \lambda_1 < \lambda_2 < 1$

$$
\int_x^\infty \int_y^\infty [\lambda_2 H(u, t) + (1 - \lambda_2)F(u)G(t) - F(u)G(t)]dt \, du
$$

$$
= \int_x^\infty \int_y^\infty \lambda_2 [H(u, t) - F(u)G(t)]dt \, du
$$

$$
\geq \int_x^\infty \int_y^\infty \lambda_1 [H(u, t) - F(u)G(t)]dt \, du
$$

$$
= \int_x^\infty \int_y^\infty [\lambda_1 H(u, t) + (1 - \lambda_1)F(u)G(t) - F(u)G(t)]dt \, du
$$
which yields
\[
\int_{-\infty}^{x} \int_{-\infty}^{y} [\lambda_2 H(u, t) + (1 - \lambda_2) F(u) G(t)] dt du \\
\ge \int_{-\infty}^{x} \int_{-\infty}^{y} [\lambda_1 H(u, t) + (1 - \lambda_1) F(u) G(t)] dt du.
\]

Thus $H_\lambda(x, y)$ is increasingly WPQD1 in $\lambda$. Similarly, $H_\lambda(x, y)$ is increasingly WPQD2 in $\lambda$ and the proof is complete.

### 3. Some Preservation Results with Application

In this section we establish preservation of the WPQD ordering under combination, mixture, transformations of random variables by increasing function, limit in distribution and other operation in statistics.

**Lemma 3.1.** Let $\underline{X} = (X_1, X_2)$ and $\underline{Y} = (Y_1, Y_2)$ have distributions $H_1$ and $H_2$, respectively, where $H_1$ and $H_2$ belong to $\beta$ such that $H_1 \succeq WPQD H_2$ and let $\underline{Z} = (Z_1, Z_2)$ with an arbitrary WPQD distribution function $H$ be independent of both $\underline{X}$ and $\underline{Y}$. Then $\underline{X} + \underline{Z} \succeq WPQD \underline{Y} + \underline{Z}$.

**Proof.** First we will show that $\underline{X} + \underline{Z}$ and $\underline{Y} + \underline{Z}$ and WPQD. For nonnegative increasing convex functions $f, g$

\[
\text{Cov}(f(X_1 + Z_1), g(X_2 + Z_2)) \\
= \text{Cov}[E\{f(X_1 + Z_1)|\underline{Z}\}, E\{g(X_2 + Z_2)|\underline{Z}\}] \\
+ E\{\text{Cov}(f(X_1 + Z_1), g(X_2 + Z_2)|\underline{Z})\}
\]

observe that according to Theorem 3 of Alzaid (1990) the second term of the right hand side of the above equation is nonnegative while the conditional expectations in the first term have the same properties in $Z_1$ and $Z_2$ as do the functions $f$ and $g$, and the first term of the right hand side is also nonnegative. Thus $\underline{X} + \underline{Z}$ is WPQD1.

A similar result holds for the WPQD2 and thus $\underline{X} + \underline{Z}$ is WPQD. Similarly, it is proved that $\underline{Y} + \underline{Z}$ is also WPQD. Next, we need show for
each \((v, w) \in \mathbb{R}^2\),

\[
\begin{align*}
(3.1) \quad & \int_v^\infty \int_w^\infty P[X_1 + Z_1 > s, X_2 + Z_2 > t] dt ds \\
& \geq \int_v^\infty \int_w^\infty P[Y_1 + Z_1 > s, Y_2 + Z_2 > t] dt ds
\end{align*}
\]

\[
\begin{align*}
(3.2) \quad & \int_{-\infty}^v \int_{-\infty}^w P[X_1 + Z_1 > s, X_2 + Z_2 > t] dt ds \\
& \geq \int_{-\infty}^v \int_{-\infty}^w P[Y_1 + Z_1 > s, Y_2 + Z_2 > t] dt ds
\end{align*}
\]

Note that

left side of (3.1)

\[
= \int_v^\infty \int_w^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty P[X_1 > s - z_1, X_2 > t - z_2] dH(z_1, z_2) dt ds
\]

\[
\geq \int_v^\infty \int_w^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty P[Y_1 > s - z_1, Y_2 > t - z_2] dH(z_1, z_2) dt ds
\]

\[
= \int_v^\infty \int_w^\infty P[Y_1 + Z_1 > s, Y_2 + Z_2 > t] dt ds = \text{ right side of (3.1)}.
\]

The above inequality follows from the assumption that \(X \geq^{WPQD} Y\).

Similarly, (3.2) is proved. Thus we complete the proof.

**Theorem 3.2.** Let \((X_i, Y_i)\) and \((U_i, V_i)\) be WPQD for \(i = 1, 2\). Let \((X_1, Y_1)\) be more WPQD than \((U_i, V_i)\) for \(i = 1, 2\). Further, let \((X_1, Y_1)\) and \((X_2, Y_2)\) be independent, \((U_1, V_1)\) and \((U_2, V_2)\) independent, and \((X_2, Y_2)\) and \((U_1, V_1)\) independent. Then

\[
(X_1 + X_2, Y_1 + Y_2) \geq^{WPQD} (U_1 + U_2, V_1 + V_2)
\]

**Proof.** By assumption, \((X_1, Y_1) \geq^{WPQD} (U_1, V_1)\). Specifying \(Z\) to be \((X_2, Y_2)\), we apply Lemma 3.1 to obtain

\[
(3.3) \quad (X_1 + X_2, Y_1 + Y_2) \geq^{WPQD} (U_1 + X_2, V_1 + Y_2).
\]
Next, we use the assumption \((X_2, Y_2) \geq WPQD(U_2, V_2)\), specify \(Z\) to be \((U_1, V_1)\), and again use Lemma 3.1, yielding

\[(3.4) \quad (U_1 + X_2, V_1 + Y_2) \geq WPQD(U_1 + U_2, V_1 + V_2).\]

By combining (3.3) and (3.4), we complete the proof according to P2. From Theorem 3 of Alzaid(1990) it follows that \((X_1, X_2)\) is more WPQD1 (WPQD2) than \((Y_1, Y_2)\) if and only if

\[(3.5) \quad \text{Cov}_{H_1}(f(X_1), g(X_2)) \geq \text{Cov}_{H_2}(f(Y_1), g(Y_2))\]

for all increasing nonnegative convex(nonpositive concave) functions \(f\) and \(g\), where \(H_1\) and \(H_2\) are the distributions of \((X_1, X_2)\) and \((Y_1, Y_2)\) respectively and \(H_1\) and \(H_2\) belong to \(\beta\).

**Theorem 3.3.** Let \(X = (X_1, X_2), Y = (Y_1, Y_2)\) have distributions \(H_1\) and \(H_2\), where \(H_1\) and \(H_2\) belong to \(\beta\) so that \((X_1, X_2) \geq WPQD1(Y_1, Y_2)\). Then \((f(X_1), X_2) \geq WPQD1(f(Y_1), Y_2)) for all increasing nonnegative convex function \(f\).

**Proof.** Let \(h\) and \(g\) be increasing, nonnegative convex functions. Since \(hf\) is an increasing nonnegative convex function for all increasing nonnegative convex function \(f\)

\[
\text{Cov}_{H_1}(h(f(X_1)), g(X_2)) \geq \text{Cov}_{H_2}(h(f(Y_1)), g(Y_2))
\]

according to (3.5). Hence \((f(X_1), X_2) \geq WPQD1(f(Y_1), Y_2)\) and the proof is complete.

**Corollary 3.4.** Let \(X = (X_1, X_2)\) and \(Y = (Y_1, Y_2)\) have distributions \(H_1\) and \(H_2\) respectively, where \(H_1\) and \(H_2\) belong to \(\beta\), such that \((X_1, X_2) \geq WPQD1(Y_1, Y_2)\). Then

\[(f(X_1), g(X_2)) \geq WPQD1(f(X_1), g(Y_2))\]

for all increasing nonnegative convex functions \(f\) and \(g\).

**Definition 3.5.** (Ebrahimi, Ghosh, 1981). A random vector \(Y\) is stochastically increasing in the random vector \(X\) if \(E(f(Y)|X=x)\) is increasing in \(x\) for all real valued increasing function \(f\). We shall use the abbreviation SI for stochastically increasing.

The following theorem gives a sufficient condition for WPQD-ness.
THEOREM 3.6. Let (a) $(X_1, X_2)$ given $\lambda$, a scalar random variable, be conditionally WPQD, and (b) $X_1$ and $X_2$ be SI in $\lambda$. Then $(X_1, X_2)$ is WPQD.

PROOF. Observe that
\[
\text{Cov}[f(X_1), g(X_2)] = E[\text{Cov}\{f(X_1), g(X_2)|\lambda\}]
\]
\[
+ \text{Cov}[E\{f(X_1)|\lambda\}, E\{g(X_2)|\lambda\}]\tag{3.6}
\]
for increasing nonnegative convex (nonpositive concave) functions $f$ and $g$. Since conditioned on $\lambda$, $(X_1, X_2)$ is WPQD1(WPQD2) by Theorem 3 of Alzaid(1990) the first term on the right hand side of (3.6) is nonnegative. From Definition 3.5 the conditional expectations in the second term on the right hand side of (3.6) are increasing functions. Since $\lambda$ is associated, according $P_4$ of Esary, et al.(1967), the covariance of the conditional expectations in the second term is nonnegative. It follows that $\text{Cov}[f(X_1), g(X_2)] \geq 0$. Thus WPQD1(WPQD2).

We may now define the class $\overline{\beta}_\lambda$ by
\[
\overline{\beta}_\lambda = \{H_\lambda : H(x, \infty|\lambda) = F(x|\lambda), H(\infty, y|\lambda) = G(y|\lambda), \]
\[
H_\lambda|\lambda \text{ is WPQD, and both } F \text{ and } G \text{ are SI in } \lambda\}
\]
The following theorem shows that if two elements of $\overline{\beta}_\lambda$ are ordered according to $\geq WPQD$, than after mixing on $\lambda$, the resulting elements in $\overline{\beta}$ preserve the same order.

THEOREM 3.7. Let $(X_1, X_2)|\lambda$ and $(Y_1, Y_2)|\lambda$ belong to $\overline{\beta}_\lambda$. Assume $(X_1, X_2)|\lambda \geq WPQD(Y_1, Y_2)|\lambda$. Then, unconditionally
\[
(X_1, X_2) \geq WPQD(Y_1, Y_2).
\]

PROOF. From Theorem 3.6 $(X_1, X_2)$ and $(Y_1, Y_2)$ are WPQD. For showing $(X_1, X_2) \geq WPQD(Y_1, Y_2)$ we have to show (3.5), i.e. $E(f(X_1), g(X_2)) \geq E(f(Y_1)g(Y_2))$ for all increasing, non-negative convex (non-positive concave) functions $f$ and $g$. Now
\[
E(f(X_1), g(X_2)) = E_\lambda\{E(f(X_1)g(X_2)|\lambda)\}
\]
\[
\geq E_\lambda\{E(f(Y_1)g(Y_2)|\lambda)\} = E(f(Y_1)g(Y_2)).
\]
Thus $(X_1, X_2) \geq WPQD(Y_1, Y_2)$ and the proof is complete.
Lemma 3.8. Suppose $(U, V) \leq WPQ^D(X, Y)$. Let $(Z_1, Z_2), (X, Y)$ and $(U, V)$ be independent random vectors and $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ increasing. Then $(f(U, Z_1), g(V, Z_2)) \leq WPQ^D(f(X, Z_1), g(Y, Z_2))$.

Proof. By the monotonicity of $f$ and $g$, the set $\{(u, a) : f(u, z_1) \leq a, g(v, z_1) \leq b\}$ is a lower rectangle and hence

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P [f(U, Z_1) \leq a, g(V, Z_2) \leq b]dbda$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P [f(U, z_1) \leq a, g(V, z_2) \leq b]dH(z_1, z_2)dbda$$

$$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P [f(X, z_1) \leq a, g(Y, z_2) \leq b]dH(z_1, z_2)dbda$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P [f(X, Z_1) \leq a, g(Y, Z_2) \leq b]dbda,$$

where $H$ is the distribution of $(Z_1, Z_2)$. Thus $(f(U, Z_1), g(V, Z_2)) \leq WPQ^{D1}(f(X, Z_1), g(Y, Z_2))$. Similarly, $(f(U, Z_1), g(V, Z_2)) \leq WPQ^{D2}(f(X, Z_1), g(Y, Z_2))$ and the proof is complete.

Theorem 3.9. Suppose $(U, V_1) \leq WPQ^D(X_1, Y_1)$ and $(U_2, V_2) \leq WPQ^D(X_2, Y_2)$. Then for independent random vectors $(X_1, Y_1), (X_2, Y_2), (U_1, V_1)(U_2, V_2)$ and for increasing functions $f$ and $g$,

$$f(U_1, U_2), g(V_1, V_2)) \leq WPQ^D(f(X_1, X_2), g(Y_1, Y_2)).$$

Proof. For increasing functions $f$ and $g$ define $f'(s, t) = f(t, s)$ and $g'(s, t) = g(t, s)$. Then $f'$ and $g'$ are also increasing. Apply Lemma 3.8 to deduce that $(f(X_1, X_2), g(Y_1, Y_2))$ is more WPQD than $(f(U_1, X_2), g(V_1, Y_2)) = (f'(X_2, U_1), g'(Y_2, V_1))$ which is more WPQD than $(f'(U_2, U_1), g'(V_2, V_1))$ which is more WPQD than $(f'(U_2, U_1), g'(V_2, V_1))$ by Lemma 3.8.

Theorem 3.10. Let $\{(X_n, Y_n) : n \geq 1\}$ and $\{(U_n, V_n) : n \geq 1\}$ be sequences of nonnegative WPQD1 bivariate random vectors with same marginals. Assume (i) $(X_n, Y_n) \geq WPQ^{D1}(U_n, V_n)$, (ii) Cov$(X_n, Y_n)$ $\rightarrow$ Cov$(X, Y)$, (iii) Cov$(U_n, V_n)$ $\rightarrow$ Cov$(U, V)$, (iv) $(X_n, Y_n)$ is weakly convergent to $(X, Y)$, (v) $(U_n, V_n)$ is weakly convergent to $(U, V)$. Then $(X, Y) \geq WPQ^{D1}(U, V)$.
Proof. Observe that
\[
\text{Cov}(X_n, Y_n) = \int_0^x \int_0^y \left[ P(X_n > s, Y_n > t) - P(X_n > s)P(Y_n > t) \right] dt ds
\]
\[
= \int_x^\infty \int_y^\infty \left[ P(X_n > s, Y_n > t) - P(X_n > s)P(Y_n > t) \right] dt ds
\]
\[
\geq \int_x^\infty \int_y^\infty \left[ P(U_n > s, V_n > t) - P(U_n > s)P(V_n > t) \right] dt ds = \text{Cov}(U_n, V_n) - \int_0^x \int_0^y \left[ P(U_n > s, V_n > t) - P(U_n > s)P(V_n > t) \right] dt ds.
\]
Taking the limit and using the dominated convergence theorem and the assumptions of theorem concerning the convergences of \(\text{Cov}(X_n, Y_n)\) and \(\text{Cov}(U_n, V_n)\) we get the required result.

Theorem 3.11. Let \((X_1, Y_1), \ldots, (X_n, Y_n)\) be independent random vectors having WPQD1 distribution \(H_1\), and let \((U_1, V_1), \ldots, (U_n, V_n)\) be independent random vectors having WPQD1 distribution \(H_2\). Let \(f\) and \(g\) be nonnegative increasing convex for the \(i\)th coordinate \((i = 1, \ldots, n)\). Assume that \(H_1 \geq \text{WPQD1} H_2\). Then
\[
\text{Cov}_{H_1}[f(X_1, \ldots, X_n), g(Y_1, \ldots, Y_n)] \\
\geq \text{Cov}_{H_2}[f(X_1, \ldots, X_n), g(Y_1, \ldots, Y_n)].
\]

Proof. According to Theorem 5 in Alzaid(1990) \((f(X_1, \ldots, X_n), g(Y_1, \ldots, Y_n))\) is WPQD1 and \((f(U_1, \ldots, U_n), g(V_1, \ldots, V_n))\) is WPQD1. Thus (3.8) holds according to (3.5).

Among the most familiar measure of dependence there are (i) Spearman's \(\rho\), \(\rho(X, Y) = 3\text{Cov}(\text{sgn}(X_2 - X_1), (\text{sgn}(Y_3 - Y_1))\), and (ii) Kendall's \(\tau\), \(\tau(X, Y) = \text{Cov}(\text{sgn}(X_2 - X_1), \text{sgn}(Y_2 - Y_1))\) where \((X_1, Y_1), (X_2, Y_2)\) and \((X_3, Y_3)\) are independent random vectors having the same distribution as \((X, Y)\). An application of Theorem 5 of Alzaid(1990) implies that all of the above measures are nonnegative under the weaker assumption that \((X, Y)\) is WPQD1. An immediate consequence of Theorem 3.11 is the following example:

Example 3.12. Let \(H_1\) and \(H_2\) be such that \(H_1 \geq \text{WPQD1} H_2\). Then Kendall's \(\tau\), and Spearman's \(\rho\) satisfy \(\tau_{H_1} \geq \tau_{H_2}\) and \(\rho_{H_1} \geq \rho_{H_2}\).
References


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