LAPLACIAN SPECTRA OF GRAPH BUNDLES

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ABSTRACT. The spectrum of the Laplacian matrix of a graph gives an information of the structure of the graph. For example, the product of non-zero eigenvalues of the characteristic polynomial of the Laplacian matrix of a graph with \( n \) vertices is \( n \) times of the number of spanning trees of that graph. The characteristic polynomial of the Laplacian matrix of a graph tells us the number of spanning trees and the connectivity of given graph. In this paper, we compute the characteristic polynomial of the Laplacian matrix of a graph bundle when its voltages lie in an abelian subgroup of the full automorphism group of the fibre; in particular, the automorphism group of the fibre is abelian. Also we study a relation between the characteristic polynomial of the Laplacian matrix of a graph \( G \) and that of the Laplacian matrix of a graph bundle over \( G \). Some applications are also discussed.

1. Introduction

Let \( G \) be a finite simple connected graph with vertex set \( V(G) = \{u_1, u_2, \ldots, u_n\} \) and edge set \( E(G) \). We denote the set of vertices adjacent to \( v \in V(G) \) by \( N(v) \) and call it the neighborhood of a vertex \( v \). Denote the degree of a vertex \( u \) by \( d(u) \). Let

\[
D(G) = \text{Diag}[d(u_1), d(u_2), \ldots, d(u_n)]
\]

be the diagonal matrix of vertex degrees. The Laplacian matrix of \( G \) is \( C(G) = D(G) - A(G) \), where \( A(G) \) is the \((0,1)\)-adjacency matrix of \( G \). The characteristic polynomial of a graph \( G \) is the characteristic polynomial \( \det(\lambda I - A(G)) \) of \( A(G) \), denoted by \( \Phi(G; \lambda) \). A zero of \( \Phi(G; \lambda) \)

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is called an eigenvalue of $G$. We denote the characteristic polynomial
$\det(\lambda - C(G))$ of the Laplacian matrix of $G$ by $\Psi(G; \lambda)$. By $|X|$, we
denote the cardinality of a finite set $X$. Convert $G$ to a digraph $\overrightarrow{G}$ by
replacing each edge $e$ of $G$ with a pair of oppositely directed edges, say $e^+$ and $e^-$. We denote the set of directed edges of $\overrightarrow{G}$ by $E(\overrightarrow{G})$. Note
that the adjacency matrix of the graph $G$ is the same as that of the
digraph $\overrightarrow{G}$. Now, we introduce the notion of a graph bundle. By $e^{-1}$
we mean the reverse edge to an edge $e \in E(\overrightarrow{G})$. Denote the directed
edge $e$ of $G$ by $uv$ if the initial and the terminal vertices of $e$ are $u$ and
$v$, respectively. For a finite group $\Gamma$, a $\Gamma$-voltage assignment of $G$ is a
function $\phi : E(\overrightarrow{G}) \rightarrow \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E(\overrightarrow{G})$.
We denote the set of all $\Gamma$-voltage assignments of $G$ by $C^1(G; \Gamma)$. Let
$F$ be another finite graph and let $\phi \in C^1(G; \text{Aut}(F))$, where $\text{Aut}(F)$
denotes the group of all graph automorphisms of $F$. Now, we construct a
new graph $G \times^\phi F$ as follows: $V(G \times^\phi F) = V(G) \times V(F)$. Two
vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \times^\phi F$ if either $u_1u_2 \in E(\overrightarrow{G})$
and $v_2 = \phi(u_1u_2)v_1$ or $u_1 = u_2$ and $v_1v_2 \in E(F)$. We call $G \times^\phi F$
the $F$-bundle over $G$ associated with $\phi$ and the first coordinate projection
$p^\phi : G \times^\phi F \rightarrow G$ the bundle projection. We also call $G$ and $F$
the base and the fibre of the bundle $G \times^\phi F$, respectively. Moreover, if
$F = \overline{K}_n$ the complement of the complete graph $K_n$ on $n$ vertices, then
an $F$-bundle over $G$ is just an $n$-fold covering graph of $G$. If $\phi(e)$ is the
identity of $\text{Aut}(F)$ for all $e \in E(\overrightarrow{G})$, then $G \times^\phi F$ is just the cartesian
product of $G$ and $F$.

2. Laplacian matrices of graph bundles

Let $F$ be a finite graph and let $\phi$ be an $\text{Aut}(F)$-voltage assignment
of $G$. For each $\gamma \in \text{Aut}(F)$, let $G_{(\phi, \gamma)}$ denote the spanning subgraph of
the digraph $\overrightarrow{G}$ whose directed edge set is $\phi^{-1}(\gamma)$, so that the digraph $\overrightarrow{G}$
is the edge-disjoint union of spanning subgraphs $G_{(\phi, \gamma)}$, $\gamma \in \text{Aut}(F)$. Let
$V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(F) = \{v_1, v_2, \ldots, v_m\}$. We define an
order relation $\leq$ on $V(G \times^\phi F)$ as follows: for any two vertices $(u_i, v_k)$
and $(u_j, v_\ell)$ of $G \times^\phi F$, $(u_i, v_k) \leq (u_j, v_\ell)$ if and only if either $k < \ell$
or $k = \ell$ and $i \leq j$. Let $P(\gamma)$ denote the $m \times m$ permutation matrix
associated with \( \gamma \in \text{Aut}(F) \) corresponding to the action of \( \text{Aut}(F) \) on \( V(F) \). Here, the tensor product \( A \otimes B \) of matrices \( A \) and \( B \) is considered as the matrix \( B \) having the element \( b_{ij} \) replaced by the matrix \( Ab_{ij} \).

It is known [6] that the adjacency matrix of a graph bundle \( G \times^\Phi F \) is

\[
A(G \times^\Phi F) = \left( \sum_{\gamma \in \text{Aut}(F)} A(\overrightarrow{G}_{(\phi, \gamma)}) \right) + I_{|V(G)|} \otimes A(F),
\]

where \( P(\gamma) \) is the permutation matrix associated with \( \gamma \) corresponding to the action of \( \text{Aut}(F) \) on \( V(F) \), and \( I_{|V(G)|} \) is the identity matrix of order \( |V(G)| \).

To find the diagonal matrix \( D(G \times^\Phi F) \) of vertex degrees, we recall that two vertices \( (u_i, v_k) \) and \( (u_j, v_\ell) \) are adjacent in \( G \times^\Phi F \) if either \( u_iu_j \in E(\overrightarrow{G}) \) and \( v_\ell = \phi(u_iu_j)v_k \) or \( u_i = u_j \) and \( v_kv_\ell \in E(F) \). Hence the degree of \( (u_i, v_k) \) is the sum of the degree of \( u_i \) in \( V(G) \) and the degree of \( v_k \) in \( V(F) \). It implies that the degree of \( (u_i, v_k) \) is \( (n(k-1) + i, n(k-1) + i) \)-entry of

\[
D(G) \otimes I_{|V(F)|} + I_{|V(G)|} \otimes D(F).
\]

That is,

\[
D(G \times^\Phi F) = D(G) \otimes I_{|V(F)|} + I_{|V(G)|} \otimes D(F).
\]

Now, the Laplacian matrix \( C(G \times^\Phi F) \) of the bundle \( G \times^\Phi F \) is given as follows.

\[
D(G \times^\Phi F) - A(G \times^\Phi F)
= \left[ D(G) \otimes I_{|V(F)|} + I_{|V(G)|} \otimes D(F) \right]
- \left[ \left( \sum_{\gamma \in \text{Aut}(F)} A(\overrightarrow{G}_{(\phi, \gamma)}) \otimes P(\gamma) \right) + I_{|V(G)|} \otimes A(F) \right]
= D(G) \otimes I_{|V(F)|}
- \left[ \left( \sum_{\gamma \in \text{Aut}(F)} A(\overrightarrow{G}_{(\phi, \gamma)}) \otimes P(\gamma) \right) + I_{|V(G)|} \otimes (A(F) - D(F)) \right]
\]
\[ D(G) \otimes I_{|V(F)|} - \left[ \left( \sum_{\gamma \in \text{Aut}(F)} A(\tilde{G}(\phi, \gamma)) \otimes P(\gamma) \right) + I_{|V(G)|} \otimes (-C(F)) \right]. \]

We summarize our discussions in the following theorem.

**Theorem 1.** The Laplacian matrix \( C(G \times^\phi F) \) of the graph bundle \( G \times^\phi F \) is

\[
D(G \times^\phi F) - A(G \times^\phi F) = D(G) \otimes I_{|V(F)|} - \sum_{\gamma \in \text{Aut}(F)} A(\tilde{G}(\phi, \gamma)) \otimes P(\gamma)
+ I_{|V(G)|} \otimes C(F).
\]

If the fibre \( F \) of the graph bundle \( G \times^\phi F \) is \( \overline{K}_n \), then \( G \times^\phi F \) is an \( n \)-fold covering graph of \( G \) and the adjacency matrix \( A(F) \) is the zero matrix. Hence, we get the following corollary.

**Corollary 1.** The Laplacian matrix \( C(G \times^\phi \overline{K}_n) \) of an \( n \)-fold covering \( G \times^\phi \overline{K}_n \) of \( G \) is

\[
D(G) \otimes I_n - \sum_{\gamma \in S_n} A(\tilde{G}(\phi, \gamma)) \otimes P(\gamma).
\]

Since the cartesian product \( G \times F \) of two graphs \( G \) and \( F \) is just the \( F \)-bundle over \( G \) associated with the trivial voltage assignment \( \phi \), i.e., \( \phi(e) = \text{the identity for all } e \in E(G) \), and \( A(G) = A(\tilde{G}) \), we get the following corollary.

**Corollary 2.** The Laplacian matrix \( C(G \times F) \) of the cartesian product \( G \times F \) of two graphs \( G \) and \( F \) is

\[
C(G) \otimes I_{|V(F)|} + I_{|V(G)|} \otimes C(F).
\]

From now on, we consider a voltage assignment \( \phi \) of \( G \) whose image lies in an abelian subgroup \( \Gamma \) of \( \text{Aut}(F) \). Since the permutation matrices \( P(\gamma), \gamma \in \Gamma \) and the Laplacian matrix \( C(F) \) of the fibre are all diagonalizable and commute with each other, they are simultaneously
diagonalizable. In other words, there exists an invertible matrix $M_{\Gamma}$ such that $M_{\Gamma}P(\gamma)M_{\Gamma}^{-1}$ and $M_{\Gamma}C(F)M_{\Gamma}^{-1}$ are diagonal matrices for all $\gamma \in \Gamma$.

For convenience, we write

$$M_{\Gamma}P(\gamma)M_{\Gamma}^{-1} = \text{Diag} \left[ \lambda_{(\gamma,1)}, \cdots, \lambda_{(\gamma,|V(F)|)} \right]$$

for $\gamma \in \Gamma$, and

$$M_{\Gamma}C(F)M_{\Gamma}^{-1} = \text{Diag} \left[ \lambda_{(F,1)}, \cdots, \lambda_{(F,|V(F)|)} \right].$$

That is, $\lambda_{(\gamma,1)}, \cdots, \lambda_{(\gamma,|V(F)|)}$ are the eigenvalues of the permutation matrix $P(\gamma)$, and $\lambda_{(F,1)}, \cdots, \lambda_{(F,|V(F)|)}$ are the eigenvalues of the Laplacian matrix $C(F)$. Then, Theorem 1 gives

$$(I_{|V(G)|} \otimes M_{\Gamma})C(G \times \phi F)(I_{|V(G)|} \otimes M_{\Gamma})^{-1}$$

$$= \bigoplus_{i=1}^{|V(F)|} \left\{ D(G) - \left( \sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\tilde{G}_{(\phi,\gamma)}) + \lambda_{(F,i)} I_{|V(G)|} \right) \right\}$$

Now, we have

**Theorem 2.** Let $\Gamma$ be an abelian subgroup of $\text{Aut}(F)$. Then, for any $\Gamma$-voltage assignment $\phi$ of $G$, the Laplacian matrix of the graph bundle $G \times \phi F$ is similar to

$$\bigoplus_{i=1}^{|V(F)|} \left\{ D(G) - \left( \sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\tilde{G}_{(\phi,\gamma)}) + \lambda_{(F,i)} I_{|V(G)|} \right) \right\}.$$

**Corollary 3.** Let $\Gamma$ be an abelian subgroup of the symmetric group $S_n$. Then, for any $\Gamma$-voltage assignment $\phi$ of $G$, the Laplacian matrix of an $n$-fold covering $G \times \phi \overline{K}_n$ of $G$ is similar to

$$\bigoplus_{i=1}^n \left\{ D(G) - \sum_{\gamma \in \Gamma} \lambda_{(\gamma,i)} A(\tilde{G}_{(\phi,\gamma)}) \right\}.$$
COROLLARY 4. The Laplacian matrix of the cartesian product $G \times F$ of two graphs $G$ and $F$ is similar to

$$\bigoplus_{i=1}^{m} \{ C(G) + \lambda_{(F,i)}I_n \}.$$ 

3. Regular coverings

A covering $p : \tilde{G} \to G$ is said to be regular if there is subgroup $A$ of the automorphism group $\text{Aut}(\tilde{G})$ of $\tilde{G}$ acting freely on $\tilde{G}$ such that $\tilde{G}/A$ is isomorphic to $G$.

The graph $G \times_{\phi} \Gamma$ derived from a voltage assignment $\phi : E(\tilde{G}) \to \Gamma$ has as its vertex set $V(G) \times \Gamma$ and as its edge set $E(G) \times \Gamma$, so that an edge of $G \times_{\phi} \Gamma$ joins a vertex $(u, \gamma)$ to $(v, \phi(e)\gamma)$ for $e = uv \in E(\tilde{G})$ and $\gamma \in \Gamma$. A vertex $(u, \gamma)$ is denoted by $u_\gamma$, and an edge $(e, \gamma)$ by $e_\gamma$. The voltage group $\Gamma$ acts on $G \times_{\phi} \Gamma$ as follows: for every $\gamma \in \Gamma$, let $\Phi_{\gamma} : G \times_{\phi} \Gamma \to G \times_{\phi} \Gamma$ denote the graph automorphism defined by $\Phi_{\gamma}(v_{\gamma'}) = v_{\gamma' \gamma^{-1}}$ on vertices and $\Phi_{\gamma}(e_{\gamma'}) = e_{\gamma' \gamma^{-1}}$ on edges. Then the natural map $G \times_{\phi} \Gamma \to (G \times_{\phi} \Gamma)/\Gamma \cong G$ is a regular $|\Gamma|$-fold covering projection.

From now on, we assume that the voltage group $\Gamma$ is a finite abelian group. Then $\Gamma$ is isomorphic to a product of cyclic groups. Say, $\Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_\ell}$. For all $\alpha = 1, \cdots, \ell$, let $\rho_{\alpha}$ denote a generator of the cyclic group $\mathbb{Z}_{n_\alpha}$ so that $\mathbb{Z}_{n_\alpha} = \{ \rho_{\alpha}^0, \rho_{\alpha}^1, \cdots, \rho_{\alpha}^{n_\alpha-1} \}$.

We define an order relation $\leq$ on $\mathbb{Z}_{n_\alpha}$ by $\rho^\ell \leq \rho^m$ if and only if $\ell \leq m$. This order relation gives the relation as in Section 2 on $\Gamma$. For any $\gamma \in \Gamma$, let $P(\gamma)$ be the permutation matrix associated with $\gamma$ under the above order. We note that the set of vertices of $G \times_{\phi} \Gamma$ also has the corresponding order relation if an order relation on $V(G)$ is given.

Chae and Lee computed the adjacency matrix of the covering graph $G \times_{\phi} \Gamma$ as follows [3]:

$$A(G \times_{\phi} \Gamma) = \sum_{(k_1, \cdots, k_\ell)} A(\tilde{G}^{\phi_{(\rho_1^{k_1}, \cdots, \rho_\ell^{k_\ell})}}) \otimes P(\rho_1^{k_1}, \cdots, \rho_\ell^{k_\ell}),$$
where \( P(\rho_1^{k_1}, \cdots, \rho_{\ell}^{k_{\ell}}) \) is the permutation matrix associated with \((\rho_1^{k_1}, \cdots, \rho_{\ell}^{k_{\ell}})\). Moreover, the adjacency matrix \( A(G \times_\phi \Gamma) \) is similar to

\[
\sum_{(k_1, \cdots, k_{\ell})} A \left( \overrightarrow{G}_{(\phi_k, \rho_1^{k_1}, \cdots, \rho_{\ell}^{k_{\ell}})} \right) \otimes \left( D(\rho_1)^{k_1} \otimes \cdots \otimes D(\rho_{\ell})^{k_{\ell}} \right),
\]

where \( D(\rho_\alpha) \) is the \( n_\alpha \times n_\alpha \) matrix

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
\zeta_\alpha & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \zeta_\alpha^{n_\alpha - 1}
\end{pmatrix}
\]

and \( \zeta_\alpha = \exp(\frac{2\pi i}{n_\alpha}) \) for \( 1 \leq \alpha \leq \ell \).

Let \( \mathbb{C} \) denote the field of complex numbers, \( \Gamma = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{\ell}} \), and let \( \phi \) be a \( \Gamma \)-voltage assignment of \( \overrightarrow{G} \). For each \((s_1, \cdots, s_{\ell}) \in \Gamma \) with \( 0 \leq s_\alpha < n_\alpha \) and \( 1 \leq \alpha \leq \ell \), we define a weight function \( \omega(s_1, \cdots, s_{\ell})(\phi) : E(\overrightarrow{G}) \to \mathbb{C} \) by

\[
\omega(s_1, \cdots, s_{\ell})(\phi)(e) = \prod_{\alpha=1}^{\ell} (\zeta_\alpha^{k_\alpha})^{s_\alpha} \quad \text{for} \quad \phi(e) = \prod_{\alpha=1}^{\ell} \rho_\alpha^{k_\alpha}.
\]

Then, we have

\[
\sum_{(k_1, \cdots, k_{\ell})} A(\overrightarrow{G}_{(\phi, \rho_1^{k_1}, \cdots, \rho_{\ell}^{k_{\ell}})}) \otimes (D(\rho_1)^{k_1} \otimes \cdots \otimes D(\rho_{\ell})^{k_{\ell}})
= \bigoplus_{(s_1, \cdots, s_{\ell})} A(\overrightarrow{G}_{\omega(s_1, \cdots, s_{\ell})(\phi)}).
\]

To find the diagonal matrix \( D(G \times_\phi \Gamma) \) of vertex degrees, we recall that an edge of \( G \times_\phi \Gamma \) joins a vertex \((u, \gamma)\) to \((v, \phi(e) \gamma)\), for \( e = uv \in E(G) \) and \( \gamma \in \Gamma \). It implies that \( D(G \times_\phi \Gamma) = D(G) \otimes I_{|\Gamma|} \). Hence, we get the following
Theorem 3. Let \( \Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t} \) and let \( \phi \) be a \( \Gamma \)-voltage assignment of \( G \). Then, the Laplacian matrix \( C(G \times \phi \Gamma) \) of a regular covering graph \( G \times \phi \Gamma \) is

\[
D(G) \otimes I_{|\Gamma|} - \sum_{(k_1, \ldots, k_t)} A(\overrightarrow{G}_{(\phi, (\rho_{k_1}, \ldots, \rho_{k_t}))}) \otimes P(\rho_{k_1}^{k_1}, \ldots, \rho_{k_t}^{k_t}).
\]

Moreover, it is similar to

\[
\bigoplus_{(s_1, \ldots, s_t)} \{D(G) - A(\overrightarrow{G}_{\omega(s_1, \ldots, s_t)}(\phi))\}.
\]

4. Computational formulas

Let \( \mathbb{C} \) denote the field of complex numbers, and let \( D \) be a digraph. A vertex-and-edge weighted digraph (in short, VEW digraph) is a pair \( D_{\omega} = (D, \omega) \), where \( \omega : E(D) \cup V(D) \to \mathbb{C} \) is a function on the set \( E(D) \) of edges in \( D \) and the set \( V(D) \) of vertices in \( D \). We call \( D \) the underlying digraph of \( D_{\omega} \) and \( \omega \) the vertex-and-edge weight function on \( D_{\omega} \). Moreover, if \( \omega(e^{-1}) = \overline{\omega(e)} \), the complex conjugate of \( \omega(e) \), for each edge \( e \in E(D) \), we say \( \omega \) is a symmetric vertex-and-edge weight function and \( D_{\omega} \) a symmetrically vertex-and-edge weighted digraph.

Given any VEW digraph \( D_{\omega} \), the adjacency matrix \( A(D_{\omega}) = (a_{ij}) \) of \( D_{\omega} \) is the square matrix of order \( |V(D)| \) defined by

\[
a_{ij} = \begin{cases} 
\omega(v_i) & \text{if } i = j, \\
\omega(v_i v_j) & \text{if } v_i v_j \in E(D), \\
0 & \text{otherwise}.
\end{cases}
\]

The characteristic polynomial of VEW digraph \( D_{\omega} \) is that of its adjacency matrix \( A(D_{\omega}) \). Now, for any \( \Gamma \)-voltage assignment \( \phi \) of \( G \), with notations as in Section 2, let \( \omega_i(\phi) : E(\overrightarrow{G}) \cup V(\overrightarrow{G}) \to \mathbb{C} \) be the function defined by \( \omega_i(\phi)(e) = -\lambda_{(\phi(e), i)} \) for \( e \in E(\overrightarrow{G}) \) and \( \omega_i(\phi)(v_j) = d(v_j) \), the degree of \( v_j \) in \( G \), so the adjacency matrix of a VEW digraph \((\overrightarrow{G}, \omega_i(\phi))\) is the matrix

\[
D(G) - \sum_{\gamma \in \Gamma} \lambda_{(\gamma, i)} A(\overrightarrow{G}_{(\phi, \gamma)});
\]
for each $i = 1, 2, \cdots, |V(F)|$. Then, we can obtain the characteristic
polynomial of the Laplacian matrix of the graph bundle $G \times^\phi F$ from
Theorem 2 as follows.

**Theorem 4.** Let $\Gamma$ be an abelian subgroup of Aut($F$) and let $\phi$ be
a $\Gamma$-voltage assignment of $G$. Then the characteristic polynomial of the
Laplacian matrix $C(G \times^\phi F)$ of $G \times^\phi F$ is

$$
\Psi(G \times^\phi F; \lambda) = \prod_{i=1}^{|V(F)|} \Phi(\overrightarrow{G}_{\omega_i(\phi)}; \lambda - \lambda_{(F,i)}).
$$

**Corollary 5.** (1) If $\Gamma$ be an abelian subgroup of $S_n$ and $\phi$ a $\Gamma$-
voltage assignment of $G$, then the characteristic polynomial of the Laplacian
matrix $C(G \times^\phi \overrightarrow{K}_n)$ of an $n$-fold covering graph of $G$ is

$$
\Psi(G \times^\phi \overrightarrow{K}_n; \lambda) = \prod_{i=1}^n \Phi(\overrightarrow{G}_{\omega_i(\phi)}; \lambda).
$$

(2) The characteristic polynomial of the Laplacian matrix $C(G \times F)$ of
the cartesian product $G \times F$ of two graphs $G$ and $F$ is

$$
\Psi(G \times F; \lambda) = \prod_{i=1}^{|V(F)|} \Psi(G; \lambda - \lambda_{(F,i)}).
$$

Corollary 5.(2) shows that the Laplacian eigenvalues of the cartesian
product $G \times F$ of graphs $G$ and $F$ are equal to all the possible
sums of eigenvalues of two factors: $\lambda_{(G,j)} + \lambda_{(F,i)}$, where $\lambda_{(G,j)}$,
$j = 1, 2, \cdots, |V(G)|$ and $\lambda_{(F,i)}$, $i = 1, 2, \cdots, |V(F)|$, are the eigenvalues
of $C(G)$ and $C(F)$, respectively.

Now, we need to calculate the characteristic polynomials $\Phi(\overrightarrow{G}_{\omega_i(\phi)}; \lambda)$
of a VEW digraph $\overrightarrow{G}_{\omega_i(\phi)}$ for $i = 1, 2, \cdots, |V(F)|$.

An undirected simple graph $S$ is called a basic figure if each of its
components is either $K_1$ or $K_2$ or a cycle $C_m (m \geq 3)$. We denote by
$B_j(G)$ the set of all subgraphs of $G$ which are basic figures with $j$ vertices.
Then, the characteristic polynomial of a VEW digraph $\overrightarrow{G}_{\omega_i(\phi)}$ is given
as follows:
Let $\Gamma$ be an abelian subgroup of $\text{Aut}(F)$. Then, for any $\Gamma$-voltage assignment $\phi$ of $G$, we have

$$
\Phi(\overrightarrow{G}, \omega_i(\phi); \lambda) = \lambda^{|V(G)|} + \sum_{j=1}^{|V(G)|} \left( \sum_{S \in B_j(G)} (-1)^{\kappa(S)} \times \prod_{u \in I_v(S)} \omega_i(\phi)(u) \right) \times \prod_{e \in K_2(S)} \omega(e^+) \omega(e^-) \times \prod_{C \in C(S)} (\omega_i(\phi)(C^+) + (\omega_i(\phi)(C^+)^{-1}) \right) \lambda^{|V(G)|-j}.
$$

In this equation, $\kappa(S)$ denotes the number of components of $S$, $K_2(S)$ the subgraph of $S$ consisting of all components isomorphic to $K_2$, $C(S)$ the set all cycle $C_m (m \geq 3)$ in $S$, and $I_v(S)$ does the set of all isolated vertices in $S$. If a component of $S$ in $G$ is a cycle $C$, $C^+$ and $C^-$ are two linear directed cycle and $\omega_i(\phi)(C^+) = \prod_{C \in E(C^+)} \omega_i(e)$. 

Now, we calculate the characteristic polynomial of a regular covering. For any $\Gamma$-voltage assignment $\phi$ of $G$, with notations as in Section 3, let $\omega_{(s_1, \ldots, s_\ell)}(\phi): E(\overrightarrow{G}) \cup V(\overrightarrow{G}) \rightarrow \mathbb{C}$ be the function defined by $\omega_{(s_1, \ldots, s_\ell)}(\phi)(e) = -\prod_{\alpha=1}^\ell (\zeta^{k_\alpha})^{s_\alpha}$ for $\phi(e) = \prod_{\alpha=1}^\ell \rho_{\alpha}^{k_\alpha}, e \in E(\overrightarrow{G})$ and $\omega_{(s_1, \ldots, s_\ell)}(\phi)(v_j) = d(v_j)$, the degree of $v_j$ in $G$.

Then, the following comes from Theorem 3.

**Theorem 5.**

$$
\Psi(G \times_\phi \Gamma; \lambda) = \Phi(C(G \times_\phi \Gamma); \lambda) = \prod_{(s_1, \ldots, s_\ell)} \Phi(\overrightarrow{G}, \omega_{(s_1, \ldots, s_\ell)}(\phi); \lambda).
$$

Now, we need to calculate the characteristic polynomial $\Phi(\overrightarrow{G}, \omega_{(s_1, \ldots, s_\ell)}(\phi); \lambda)$ of a VEW digraph $\overrightarrow{G}, \omega_{(s_1, \ldots, s_\ell)}(\phi)$.

Finally, we compute the characteristic polynomial $\Phi(\overrightarrow{G}, \omega_{(s_1, \ldots, s_\ell)}(\phi); \lambda)$ of a VEW digraph $\overrightarrow{G}, \omega_{(s_1, \ldots, s_\ell)}(\phi)$ for a pseudograph $G$ as a generalization.

In an undirected pseudograph, two elementary configurations $S_1$ and $S_2$ are equivalent if the identity map of vertex set $V(G)$ induces an isomorphism between $S_1$ and $S_2$. We denote the set of equivalence classes
of $B_j(G)$ by $[B_j(G)]$ for $j = 1, \cdots, |V(G)|$. Let $[S]$ be an element of $[B_j(G)]$. Then $[S]$ is an equivalence class of $K_1$ or $K_2$ or cycles. Let $E(K_2[S])$ be the equivalence classes of the copies of $K_2$ and $E(C[S])$ the equivalence classes of the cycles in $[S]$. Note every copy of $K_2$ in $G$ induces two directed edges in $\overrightarrow{G}$, say $e^+$ and $e^-$, and every loop is a cycle of length 1. Then we can get the following theorem.

**Theorem 6.** Let $\Gamma$ be a finite abelian group and let $\phi$ be a $\Gamma$-voltage assignment of $G$. Let $\omega$ be one of $\omega_{(s_1, \cdots, s_t)}(\phi)$. Then, for each $[S] \in [B_j(G)]$, the contribution of $[S]$ in the coefficient of $\lambda^{|V(G)|-j}$ of $\Phi(\overrightarrow{G}; \omega, \lambda)$ is

\[
(-1)^{\kappa(S)} \prod_{u \in J_\omega(S)} \omega(u) \prod_{[e] \in E(K_2[S])} \left( \sum_{e \in [e]} \omega(e^+) \right) \left( \sum_{e \in [e]} (\omega(e^+))^{-1} \right) 2^{|E(C[S])|} \times \prod_{[C] \in E(C[S])} \left( \sum_{C \in [C]} \text{Re}(\omega(C^+)) \right),
\]

where $\text{Re}(\omega(C^+))$ is the real part of $\prod_{e \in C^+} \omega(e)$ and $S$ is a representative of $[S]$.

5. Applications

Let $n$ be a positive integer. The wrapped butterfly $WB_n$ of order $n$ has vertex set

\[ V(WB_n) = \mathbb{Z}_n \times \mathbb{Z}_2^n, \]

and each vertex

\[ < \ell, \beta_0 \beta_1 \cdots \beta_{\ell-1} \alpha \beta_{\ell+1} \cdots \beta_{n-1} > \]

is adjacent to each of the vertices

\[ < \ell + 1 \, (\mod n), \beta_0 \beta_1 \cdots \beta_{\ell-1} \omega \beta_{\ell+1} \cdots \beta_{n-1} > \]
for $\omega \in \mathbb{Z}_2$. For example, $WB_3$ can be drawn as follows:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{wrapped_butterfly}
\caption{The wrapped butterfly $WB_3$.}
\end{figure}

Let $WG_n$ be the pseudograph with vertex set $V(WG_n) = \{v_0, \cdots, v_{n-1}\}$, edge set $E(WG_n) = \{d_1, \cdots, d_n; e_1, \cdots, e_n\}$, where both $d_i$ and $e_i$ having the same endpoints $v_{i-1}$ and $v_i$ for $1 \leq i \leq n$.

It is known [7] that a wrapped butterfly $WB_n$ can be represented as a covering graph $WG_n \times_\phi \mathbb{Z}_2^n$ with a $\mathbb{Z}_2^n$-voltage assignment $\phi; \phi(e_i) = 0 \cdots 0$ for all $i = 1, 2, \cdots, n$ and $\phi(d_1) = 10 \cdots 0, \cdots, \phi(d_n) = 0 \cdots 1$. Then $\omega(s_1, \cdots, s_n)(\phi)(e_i) = -1$ for all $i = 1, 2, \cdots, n$ and

$$
\omega(s_1, \cdots, s_n)(\phi)(d_i) = \begin{cases} 
1 & \text{if } s_i = 1, \\
-1 & \text{otherwise,}
\end{cases}
$$

and

$$
\omega(s_1, \cdots, s_n)(\phi)(v_i) = 4
$$

for all $i = 1, 2, \cdots, n$.

For example, if $G = WG_3$ and $(s_1, s_2, s_3) = (1, 1, 0)$, then we get the following figures.
The adjacency matrix $A(\overrightarrow{G}_{\omega(1,1,0)}(\phi))$ is
\[
\begin{pmatrix}
4 & 0 & 0 \\
0 & 4 & -2 \\
0 & -2 & 4
\end{pmatrix}.
\]

Hence, we have
\[
A(\overrightarrow{G}_{\omega(1,1,0)}(\phi)) = (4) \oplus \begin{pmatrix}
4 & -2 \\
-2 & 4
\end{pmatrix}
\]
\[
= (4D(P_1) - 2A(P_1)) \oplus (4D(P_2) - 2A(P_2)),
\]
and
\[
\Phi(\overrightarrow{G}_{\omega(1,1,0)}(\phi); \lambda) = \det((\lambda - 4)I_1 + 2A(P_1))\det((\lambda - 4)I_2 + 2A(P_2))
\]
\[
= (-1)^32^3 \det\left(\frac{4-\lambda}{2}I_1 - A(P_1)\right) \det\left(\frac{4-\lambda}{2}I_2 - A(P_2)\right)
\]
\[
= (-1)^32^3 \Phi\left(P_1; \frac{4-\lambda}{2}\right) \Phi\left(P_2; \frac{4-\lambda}{2}\right).
\]

In general, if $P_k$ is a path on $k$ vertices and $C_k$ is a cycle of length $k$, then
\[
\Phi(WG_{n\omega(s_1,\ldots,s_n)}(\phi); \lambda)
\]
\[
= \begin{cases}
(-1)^n2^n\Phi(C_n; \frac{4-\lambda}{2}) & \text{if } (s_1,\ldots,s_n) = (0,\ldots,0), \\
(-1)^n2^n\Phi(P_{k_1}; \frac{4-\lambda}{2}) \cdots \Phi(P_{k_r}; \frac{4-\lambda}{2}) & \text{otherwise},
\end{cases}
\]
where $\{k_1,\ldots,k_r\} \subset \{1,\ldots,n\}$.

Let $1 \leq r \leq n - 1$. Identify $d_i$ with $d_j$ and $s_i$ with $s_j$ if $i \equiv j$ (mod $n$).
Then $\Phi(P_r; \frac{4-\lambda}{2})$ is a factor of $\Phi(WG_{n\omega(s_1,\ldots,s_n)}(\phi); \lambda)$ if and only if
\[
\omega(s_1,\ldots,s_n)(\phi)(d_{i+1}) = \cdots = \omega(s_1,\ldots,s_n)(\phi)(d_{i+r-1}) = -1
\]
and
\[
\omega(s_1,\ldots,s_n)(\phi)(d_i) = \omega(s_1,\ldots,s_n)(\phi)(s_{i+r}) = 1
\]
for some $i \in \{1,2,\ldots,n\}$. This is equivalent to say that
\[
s_{i+1} = \cdots = s_{i+r-1} = 0, \quad s_i = s_{i+r} = 1
\]
for some \( i \in \{1, 2, \cdots, n\} \). Since

\[
\left\{(s_1, \cdots, s_n) \in \mathbb{Z}_2^n \mid s_{i+1} = \cdots = s_{i+r-1} = 0, s_i = s_{i+r} = 1\right\} = 2^{n-r-1}
\]

for each \( 1 \leq i \leq n \), the multiplicity of \( \Phi(P_r; \frac{4-\lambda}{2}) \) in \( \Phi(C(WB_n); \lambda) \) is \( n \cdot 2^{n-r-1} \).

Now, \( \Phi(P_n; \frac{4-\lambda}{2}) \) is a factor of \( \Phi(WG_{n_{w(s_1, \cdots, s_n), \phi}}; \lambda) \) if and only if for some \( i \in \{1, 2, \cdots, n\} \), \( \omega(s_1, \cdots, s_n)(\phi)(d_i) = 1 \) and \( \omega(s_{i+1}, \cdots, s_n)(\phi)(d_k) = -1 \) for all \( k \neq i \). Hence the multiplicity of \( \Phi(P_n; \frac{4-\lambda}{2}) \) in \( \Phi(C(WB_n); \lambda) \) is \( n \).

Clearly, there exists only one factor of \( \Phi(C_n; \frac{4-\lambda}{2}) \) in \( \Phi(C(WB_n); \lambda) \). Therefore

\[
\Psi(WB_n; \lambda) = \Phi(C(WB_n); \lambda) = \prod_{(s_1, \cdots, s_n)} \Phi(WG_{n_{w(s_1, \cdots, s_n), \phi}}; \lambda)
\]

\[
= ((-1)^n 2^n)^{2^{n-1}} \prod_{r=1}^{n-1} \Phi \left( P_r; \frac{4-\lambda}{2} \right) \Phi \left( C_n; \frac{4-\lambda}{2} \right).
\]

To get the number of all spanning trees of \( WB_n \), we need to calculate the product of all non-zero roots of both

\[
\Phi \left( P_r; \frac{4-\lambda}{2} \right) = 0 \quad \text{and} \quad \Phi \left( C_n; \frac{4-\lambda}{2} \right) = 0.
\]

Since \( \exp \left( \frac{2\pi i}{n} \right) \) is the root of the equation \( x^n - 1 = 0 \),

\[
\left\{ 1 - \exp \left( \frac{2\pi i}{n} \right) \right\} \times \cdots \times \left\{ 1 - \exp \left( \frac{2(n-1)\pi i}{n} \right) \right\} = n.
\]

If \( 1 \leq k \leq n - 1 \),

\[
1 - \exp \left( \frac{2k\pi}{n} i \right) = \exp \left( \frac{k\pi}{n} i \right) \left\{ \exp \left( \frac{-k\pi}{n} i \right) - \exp \left( \frac{k\pi}{n} i \right) \right\}
\]

\[
= (-2i) \exp \left( \frac{k\pi}{n} i \right) \sin \frac{k\pi}{n}.
\]
Hence,
\[ \prod_{k=1}^{n-1} \sin^2 \left( \frac{k\pi}{n} \right) = \frac{n^2}{4^{n-1}}. \]

The spectrum of a path \( P_n \) consists of the numbers \( 2 \cos \frac{k\pi}{n+1} (k = 1, \cdots, n) \). Put \( \frac{4-\lambda}{2} = 2 \cos \frac{k\pi}{n+1} \). Then \( \lambda = 4(1 - \cos \frac{k\pi}{n+1}) \neq 0 \) for \( k = 1, \cdots, n \) and
\[
\prod_{k=1}^{n} 4 \left( 1 - \cos \frac{k\pi}{n+1} \right) = 4^n \prod_{k=1}^{n} \left( 1 - \cos \frac{k\pi}{n+1} \right) \\
= 4^n \left( \prod_{k=1}^{n} \left( 1 - \cos \frac{k\pi}{n+1} \right) \right)^{\frac{1}{2}} \\
= 4^n \left( \prod_{k=1}^{n} \left( 1 - \cos \frac{k\pi}{n+1} \right) \left( 1 - \cos \frac{(n+1) - k\pi}{n+1} \right) \right)^{\frac{1}{2}} \\
= 4^n \left( \prod_{k=1}^{n} \sin^2 \frac{k\pi}{n+1} \right)^{\frac{1}{2}} \\
= (n + 1)2^n.
\]

The spectrum of a cycle \( C_n \) consists of the numbers \( 2 \cos \frac{2k\pi}{n} (k = 1, \cdots, n) \).
From \( \frac{4-\lambda}{2} = 2 \cos \frac{2k\pi}{n} \), we can get \( \lambda = 4 - 4 \cos \frac{2k\pi}{n} \neq 0 \) for \( k = 1, \cdots, n - 1 \). Hence
\[
\prod_{k=1}^{n-1} \left( 4 - 4 \cos \frac{2k\pi}{n} \right) = 4^{n-1} \prod_{k=1}^{n-1} \left( 1 - \cos \frac{2k\pi}{n} \right) \\
= 4^{n-1} \prod_{k=1}^{n-1} \left( 2 \sin^2 \frac{2k\pi}{n} \right) \\
= 4^{n-1} 2^{n-1} \prod_{k=1}^{n-1} \sin^2 \frac{2k\pi}{n} \\
= n^2 \cdot 2^{n-1}.
\]

We summarize our discussions in the following theorem.
THEOREM 7. The number $t(WB_n)$ of spanning trees of the wrapped butterfly $WB_n$ is $n(n+1)^n2^{n^2-1+n2^n} \sum_{r=1}^{n-1} r^{2-r-1} \prod_{r=1}^{n-1} (r+1)^{n \cdot 2^{n-r-1}}$.

PROOF. Let $t(G)$ denote the number of spanning trees contained in a graph $G$. Then it is well known that $t(G) = \frac{1}{n} \prod \lambda$, where $\lambda$ runs through all non-zero eigenvalues of the Laplacian matrix of $G$. Hence

$$t(WB_n) = \frac{1}{n \cdot 2^n} \prod_{r=1}^{n-1} ((r+1)2^r)^{n^{2^n-r-1}} ((n+1) \cdot 2^n)^{n^2 \cdot 2^{n-1}}$$

$$= n(n+1)^n2^{n^2-1+n2^n} \sum_{r=1}^{n-1} r^{2-r-1} \prod_{r=1}^{n-1} (r+1)^{n \cdot 2^{n-r-1}}.$$

References


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