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An Alternative Proof of the Asymptotic Behavior of GLSE in Polynomial MEM

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Abstract

Polynomial measurement error model(MEM) with one predictor is considered. It is briefly mentioned that Chan and Mak's generalized least squares estimator(GLSE) can be derived more easily if Hermite polynomial concept is applied. It is proved that GLSE derived using new procedure is equivalent to the estimator obtained from corrected score function. Finally, much simpler proof of the asymptotic behavior of GLSE than that of Chan and Mak is provided. Much simpler formula of asymptotic covariance matrix of GLSE is a part of that proof.

1. Introduction

Generalized least squares estimation method is one of estimation schemes that can be applied to polynomial MEM. Chan and Mak(1985) derive a GLSE for the polynomial functional relationship(Fuller 1987) and show that it follows an asymptotic normal distribution. But their derivation procedure is complicated and, as a result, asymptotic covariance matrix formula is too clumsy. Use of Hermite polynomial concepts, which is introduced in Moon and Gunst(1995), makes it possible to avoid that kind of difficulties since the expectation of the standardized m-th Hermite polynomial(see section 2) of normal variate is equal to the m-th power of its mean. A topic of this paper is a simpler derivation and proof of asymptotic behavior(including asymptotic covariance formula) of polynomial GLSE. We can show the equivalence of GLSE and Stefanski(1989) & Nakamura(1990)'s corrected score function estimator as a by-product of simpler procedure that utilizes the concept of Hermite polynomial.

In section 2, some basic notation and polynomial functional model is defined. Brief comments on the derived GLSE and on the equivalence relation mentioned above are also included. Derivation and proof of asymptotic behavior of GLSE are topic of the final section.

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2. Definitions and GLS Estimator

The k-th order polynomial functional relationship with single predictor is given by

$$\begin{aligned} \psi_i &= \beta_0 + \beta_1 \pi_i + \beta_2 \pi_i^2 + \cdots + \beta_k \pi_i^k \\ &= \boldsymbol{\pi}_i^{\,t} \boldsymbol{\beta}, \qquad\qquad i = 1, 2, 3, \cdots, n, \end{aligned} \tag{2.1}$$

where

$$\boldsymbol{\pi}_i = (1, \pi_i, \pi_i^2, \cdots, \pi_i^k)^t,$$

$$\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2, \cdots, \beta_k)^t.$$

Bold-faced letters denote vectors or matrices and all vectors are column ones. Let $\boldsymbol{z}_i = (y_i, x_i)^t$ denote the vector of observable response and predictor variables. That is, $\boldsymbol{z}_i = \boldsymbol{\xi}_i + \boldsymbol{w}_i$ with $\boldsymbol{\xi}_i = (\psi_i, \pi_i)^t$ denoting the vector of error-free variates and $\boldsymbol{w}_i = (v_i, u_i)^t$ the vector of measurement errors. The vector of measurement errors $\boldsymbol{w}_i$ are assumed to be i.i.d. N($\boldsymbol{0}$, $\boldsymbol{\Sigma}_{ww}$) with completely known covariance matrix $\boldsymbol{\Sigma}_{ww} = \begin{pmatrix} \sigma_{vv} & \sigma_{vu} \\ \sigma_{uv} & \sigma_{uu} \end{pmatrix}$.

Let $H_m(z)$ be the m-th Hermite polynomial and let $P_m(z) = \sigma^m H_m(\frac{z}{\sigma})$ be the standardized m-th Hermite polynomial of z, where var(z) $= \sigma^2$. Finally, let $\boldsymbol{p}_i$ denote a vector of standardized m-th(m = 1, 2, 3, $\cdots$, k) Hermite polynomials in x_i and let $\boldsymbol{f}_i$ be the vector of deviations from the corresponding powers in the error-free predictor. That is,

$$\boldsymbol{p}_i = \boldsymbol{\pi}_i + \boldsymbol{f}_i = (1, P_1(x_i), P_2(x_i), \cdots, P_k(x_i))^t,$$

$$\boldsymbol{f}_i = (0, P_1(x_i) - \pi_i, P_2(x_i) - \pi_i^2, \cdots, P_k(x_i) - \pi_i^k)^t.$$

With the notation and definitions given above, much simpler GLS estimation procedure than that of Chan and Mak is provided in Moon and Gunst. The resultant GLSE and some necessary notation are restated in the below briefly since we need to mention them in deriving the asymptotic behavior of GLSE.

$$\hat{\boldsymbol{\beta}} = \overline{\boldsymbol{H}}_2^{\,-1} \, \overline{\boldsymbol{H}}_1 \begin{pmatrix} \sigma^{vv} \\ \sigma^{uv} \end{pmatrix} / \sigma^{vv} - \begin{pmatrix} 0 \\ \sigma^{uv} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} / \sigma^{vv}, \tag{2.2}$$

where $\overline{H}_1$ and $\overline{H}_2$ are averages of H_{1i}'s and H_{2i}'s respectively, with

$$H_{1i} = p_i z_i^t - \hat{E}(p_i w_i^t) = p_i z_i^t - \begin{pmatrix} 0 & 0 \\ \sigma_{vu} & \sigma_{uu} \\ 2\sigma_{vu}P_1(x_i) & 2\sigma_{uu}P_1(x_i) \\ 3\sigma_{vu}P_2(x_i) & 3\sigma_{uu}P_2(x_i) \\ \vdots & \vdots \\ k\sigma_{vu}P_{k-1}(x_i) & k\sigma_{uu}P_{k-1}(x_i) \end{pmatrix},$$

$$H_{2i} = \begin{pmatrix} 1 & P_1(x_i) & P_2(x_i) & \cdots & P_k(x_i) \\ P_1(x_i) & P_2(x_i) & P_3(x_i) & \cdots & P_{k+1}(x_i) \\ \vdots & \vdots & \vdots & & \vdots \\ P_k(x_i) & P_{k+1}(x_i) & P_{k+2}(x_i) & \cdots & P_{2k}(x_i) \end{pmatrix}, \quad i = 1, 2, 3, \cdots, n, \quad \text{and}$$

$$\Sigma_{ww}^{-1} = \begin{pmatrix} \sigma^{vv} & \sigma^{vu} \\ \sigma^{uv} & \sigma^{uu} \end{pmatrix}.$$

In the following theorem, it is shown that GLSE of polynomial MEM given in (2.2) is equivalent to the estimator obtained(assuming $\sigma_{uv} = 0$) from corrected score function, which is provided in Moon and Gunst as $[n^{-1} \sum_{i=1}^{n} H_{2i}]^{-1} [n^{-1} \sum_{i=1}^{n} p_i y_i]$.

Theorem 1. Suppose that model (2.1) holds with $\sigma_{uv} = 0$. Then, (2.2) is equivalent to the estimator derived using corrected score function.

Proof: Since $\sigma_{uv} = \sigma^{uv} = 0$, we have from (2.2),

$$\hat{\beta} = \overline{H}_2^{-1} \frac{1}{n} \sum_{i=1}^{n} [p_i z_i^t - \hat{E}(p_i w_i^t)] \binom{1}{0}$$

$$= \overline{H}_2^{-1} \frac{1}{n} \sum_{i=1}^{n} [\begin{pmatrix} 1 \\ P_1(x_i) \\ P_2(x_i) \\ \vdots \\ P_k(x_i) \end{pmatrix} (y_i \ \ x_i) - \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{uu} \\ 0 & 2\sigma_{uu}P_1(x_i) \\ \vdots & \vdots \\ 0 & k\sigma_{uu}P_{k-1}(x_i) \end{pmatrix}] \binom{1}{0}$$

$$= \overline{H}_2^{-1} \frac{1}{n} \sum_{i=1}^{n} p_i y_i,$$

which reduces to Nakamura and Stefanski's corrected score function estimator. □

3. Asymptotic Behavior of GLSE

The limiting distribution of $\hat{\boldsymbol{\beta}}$ is given in Theorem 2. Before giving that theorem, one lemma which is useful in the proof of theorem is provided.

Lemma. Suppose that model (2.1) holds. Then, with $\boldsymbol{Y}_{xx} = n^{-1} \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i^t$,

$$\boldsymbol{\beta} = \boldsymbol{Y}_{xx}^{-1} E(\overline{\boldsymbol{H}}_1)\begin{pmatrix}\sigma^{vv}\\ \sigma^{uv}\end{pmatrix} / \sigma^{vv} - \begin{pmatrix}0\\ \sigma^{uv}\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{pmatrix} / \sigma^{vv}. \tag{3.1}$$

Proof: From $\phi_i = \boldsymbol{x}_i^t \boldsymbol{\beta}$, $i = 1, 2, 3, \cdots, n$, it follows that

$$\begin{aligned}\boldsymbol{\beta} &= \left[n^{-1}\sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i^t\right]^{-1}\left[n^{-1}\sum_{i=1}^{n} \boldsymbol{x}_i \phi_i\right]\\ &= \boldsymbol{Y}_{xx}^{-1} E(\overline{\boldsymbol{H}}_1)\begin{pmatrix}1\\0\end{pmatrix}.\end{aligned} \tag{3.2}$$

Hence, it is sufficient to show that (3.1) is equivalent to (3.2).

$$\begin{aligned}&\boldsymbol{Y}_{xx}^{-1} E(\overline{\boldsymbol{H}}_1)\begin{pmatrix}\sigma^{vv}\\ \sigma^{uv}\end{pmatrix} / \sigma^{vv} - \begin{pmatrix}0\\ \sigma^{uv}\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{pmatrix} / \sigma^{vv}\\ &= \boldsymbol{Y}_{xx}^{-1} E(\overline{\boldsymbol{H}}_1)\begin{pmatrix}1\\0\end{pmatrix} + \boldsymbol{Y}_{xx}^{-1} E(\overline{\boldsymbol{H}}_1)\begin{pmatrix}0\\ \sigma^{uv}\end{pmatrix} / \sigma^{vv} - \begin{pmatrix}0\\ \sigma^{uv}\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{pmatrix} / \sigma^{vv}\\ &= \boldsymbol{\beta},\end{aligned}$$

given in (3.2), since the second columns of $E(\overline{\boldsymbol{H}}_1)$ and $\boldsymbol{Y}_{xx}$ are the same. □

Theorem 2. Suppose that model (2.1) holds. Assume the followings:

(a) $\boldsymbol{Y}_{xx}$ is a positive definite matrix for all $n > k+1$ and

$$\lim_{n\to\infty} \boldsymbol{Y}_{xx} = \boldsymbol{\Gamma}_{xx},$$

where $\boldsymbol{\Gamma}_{xx}$ is positive definite.

(b) $n^{-1}\sum_{i=1}^{n} |\pi_i|^j$ converges for $j = 2k+1$ to $4k+\delta$, where $\delta > 0$.

Then, $n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is asymptotically normal with mean $\mathbf{0}$ and covariance matrix

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} E(\boldsymbol{\phi}_i \boldsymbol{\phi}_i^t),$$

where $\hat{\boldsymbol{\beta}}$ is given in (2.2) and

$$\boldsymbol{\phi}_i = \boldsymbol{Y}_{xx}^{-1} [\, \boldsymbol{H}_{1i} - E(\overline{\boldsymbol{H}}_1)] \begin{pmatrix} \sigma^{vv} \\ \sigma^{uv} \end{pmatrix} / \sigma^{vv} - \boldsymbol{Y}_{xx}^{-1} (\, \boldsymbol{H}_{2i} - \boldsymbol{Y}_{xx}) \boldsymbol{Y}_{xx}^{-1} E(\overline{\boldsymbol{H}}_1) \begin{pmatrix} \sigma^{vv} \\ \sigma^{uv} \end{pmatrix} / \sigma^{vv}$$

$$= \boldsymbol{Y}_{xx}^{-1} [\, \boldsymbol{H}_{1i} - \boldsymbol{H}_{2i} \boldsymbol{Y}_{xx}^{-1} E(\overline{\boldsymbol{H}}_1)] \begin{pmatrix} \sigma^{vv} \\ \sigma^{uv} \end{pmatrix} / \sigma^{vv}.$$

Proof: From (2.2),

$$\hat{\boldsymbol{\beta}} = [\, \{\, \overline{\boldsymbol{H}}_2 - E(\overline{\boldsymbol{H}}_2)\,\} + E(\overline{\boldsymbol{H}}_2)\,]^{-1}\, [\, \{\, \overline{\boldsymbol{H}}_1 - E(\overline{\boldsymbol{H}}_1)\,\} + E(\overline{\boldsymbol{H}}_1)\,] \begin{pmatrix} \sigma^{vv} \\ \sigma^{uv} \end{pmatrix} / \sigma^{vv} - \begin{pmatrix} 0 \\ \sigma^{uv} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} / \sigma^{vv}.$$

Since $\overline{\boldsymbol{H}}_l - E(\overline{\boldsymbol{H}}_l) = O_p(n^{-1/2})$, $l = 1, 2$, and since $E(\overline{\boldsymbol{H}}_2) = \boldsymbol{Y}_{xx}$ is positive definite by assumption (a),

$$\hat{\boldsymbol{\beta}} = \left[\, \boldsymbol{Y}_{xx}^{-1} - \boldsymbol{Y}_{xx}^{-1} (\, \overline{\boldsymbol{H}}_2 - \boldsymbol{Y}_{xx}) \boldsymbol{Y}_{xx}^{-1} + O_p(n^{-1}) \right]$$

$$\times\, [\, \{\, \overline{\boldsymbol{H}}_1 - E(\overline{\boldsymbol{H}}_1)\,\} + E(\overline{\boldsymbol{H}}_1)\,] \begin{pmatrix} \sigma^{vv} \\ \sigma^{uv} \end{pmatrix} / \sigma^{vv} - \begin{pmatrix} 0 \\ \sigma^{uv} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} / \sigma^{vv}.$$

$$= Y_{xx}^{-1} E(\overline{H}_1)\begin{pmatrix}\sigma^{vv}\\ \sigma^{uv}\end{pmatrix}/\sigma^{vv} + Y_{xx}^{-1}[\overline{H}_1 - E(\overline{H}_1)]\begin{pmatrix}\sigma^{vv}\\ \sigma^{uv}\end{pmatrix}/\sigma^{vv}$$

$$- Y_{xx}^{-1}(\overline{H}_2 - Y_{xx}) Y_{xx}^{-1} E(\overline{H}_1)\begin{pmatrix}\sigma^{vv}\\ \sigma^{uv}\end{pmatrix}/\sigma^{vv} - \begin{pmatrix}0\\ \sigma^{uv}\\ 0\\ \cdot\\ \vdots\\ 0\end{pmatrix}/\sigma^{vv} + O_p(n^{-1}).$$

Therefore by lemma,

$$\hat{\beta} - \beta = Y_{xx}^{-1}\{\overline{H}_1 - E(\overline{H}_1)\}\begin{pmatrix}\sigma^{vv}\\ \sigma^{uv}\end{pmatrix}/\sigma^{vv} - Y_{xx}^{-1}(\overline{H}_2 - Y_{xx}) Y_{xx}^{-1} E(\overline{H}_1)\begin{pmatrix}\sigma^{vv}\\ \sigma^{uv}\end{pmatrix}/\sigma^{vv} + O_p(n^{-1})$$

$$= \frac{1}{n}\sum_{i=1}^{n} \phi_i + O_p(n^{-1}). \tag{3.3}$$

Multiplying by $n^{1/2}$ on each side of (3.3) results in

$$n^{1/2}(\hat{\beta} - \beta) = n^{-1/2}\sum_{i=1}^{n} \phi_i + O_p(n^{-1/2}).$$

Consider the distribution of $n^{-1/2}\sum_{i=1}^{n} \lambda^t \phi_i$, where λ is an arbitrary nonzero $(k+1)$ x 1 vector. By the assumption made on w_i, it is evident that

$$E[(\lambda^t \phi_i)^2] = \lambda^t E(\phi_i \phi_i^t)\lambda < \infty \text{ and } E[|\lambda^t \phi_i|^{2+\delta}] < \infty, \text{ where } \delta > 0.$$

And, since $E(\phi_i \phi_i^t)$ contains terms from π_i to π_i^{4k},

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} E[|\lambda^t \phi_i|^{2+\delta}]}{[\sum_{i=1}^{n} E(\lambda^t \phi_i)^2]^{(2+\delta)/2}} = 0,$$

by assumptions (a) and (b). Therefore, we have

$$\frac{n^{-1/2}\sum_{i=1}^{n}\lambda^t\phi_i}{[\lambda^t E(n^{-1}\sum_{i=1}^{n}\phi_i\phi_i^t)\lambda]^{1/2}} \xrightarrow{d} N(0,\ 1).$$

The result of the theorem follows from Slusky's theorem. □

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