A CHARACTERIZATION OF
MCSHANE INTEGRABILITY

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Abstract. In this paper we prove that for functions from \([0,1]\)
into a totally ordered AL-space, Mcshane integrability and absolute
Mcshane integrability are equivalent.

1. Introduction

In 1990 Gordon[4] introduced the Mcshane integral of Banach-valued functions. This integral is a generalized Riemann integral of
functions which have values in a Banach space. For real-valued func-
tions the Mcshane integral and the Lebesgue integral are equivalent.
Gordon[4] and Fremlin and Mendoza[2] have developed the prop-
ties of this integral. A Bochner integrable function is Mcshane inte-
grable [4], and a Mcshane integrable function is Pettis integrable [2].
Many authors have studied the Bochner integral and the Pettis integral
([1],[3],[5],[6]).

In this paper we prove that for functions from \([0,1]\) into a totally
ordered AL-space, Mcshane integrability and absolute Mcshane inte-
grability are equivalent.

2. Preliminaries

Unless otherwise stated, we always assume that \(X\) is a real Banach
space with dual \(X^*\) and \(([0,1], \Sigma, \mu)\) is the Lebesgue measure space.

Gordon[4] introduced the Mcshane integral of Banach-valued func-
tions.
Definition 2.1. A McShane partition of $[0,1]$ is a finite collection $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ such that $\{[a_i, b_i] : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[0,1]$ covering $[0,1]$ and $t_i \in [0,1]$ for each $i \leq n$. A gauge on $[0,1]$ is a function $\delta : [0,1] \to (0, \infty)$. A McShane partition $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ is subordinate to a gauge $\delta$ if $[a_i, b_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for every $i \leq n$. If $f : [0,1] \to X$ and if $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[0,1]$, we will denote $\int f(\mathcal{P})$ for $\sum_{i=1}^{n} f(t_i)(b_i - a_i)$. A function $f : [0,1] \to X$ is McShane integrable on $[0,1]$, with McShane integral $\omega$, if for every $\varepsilon > 0$ there exists a gauge $\delta : [0,1] \to (0, \infty)$ such that $\|\omega - f(\mathcal{P})\| < \varepsilon$ for every McShane partition $\mathcal{P} = \{([a_i, b_i], t_i) : 1 \leq i \leq n\}$ of $[0,1]$ subordinate to $\delta$.

Gorden [4] obtained the following theorem which is useful to prove our result.

Theorem 2.2 [4]. The function $f : [0,1] \to X$ is McShane integrable on $[0,1]$ if and only if for each $\varepsilon > 0$ there exists a gauge $\delta$ on $[0,1]$ such that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \varepsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are McShane partitions of $[0,1]$ subordinate to $\delta$.

Definition 2.3. A function $f : [0,1] \to X$ is absolutely McShane integrable on $[0,1]$ if for each $\varepsilon > 0$ there exists a gauge $\delta$ on $[0,1]$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{h} ||f(t_i') - f(t_j'')|| \mu([a_i', b_i'] \cap [a_j'', b_j'']) < \varepsilon$$

whenever $\mathcal{P}' = \{([a_i', b_i'], t_i') : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{([a_j'', b_j''], t_j'') : 1 \leq j \leq h\}$ are McShane partitions of $[0,1]$ subordinate to $\delta$.

3. Main Result

In this section, we give a characterization of McShane integrability in terms of absolute McShane integrability.

Lemma 3.1. $f : [0,1] \to X$ is McShane integrable if and only if for each $\varepsilon > 0$ there exists a gauge $\delta$ on $[0,1]$ such that

$$\|\sum_{i=1}^{k} \sum_{j=1}^{h} [f(t_i') - f(t_j'')] \mu([a_i', b_i'] \cap [a_j'', b_j''])\| < \varepsilon$$
whenever $\mathcal{P}' = \{(a'_i, b'_i], t'_i) : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{(a''_j, b''_j), t''_j) : 1 \leq j \leq h\}$ are Mcshane partitions of $[0, 1]$ subordinate to $\delta$.

Proof. Let $\mathcal{P}' = \{(a'_i, b'_i], t'_i) : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{(a''_j, b''_j), t''_j) : 1 \leq j \leq h\}$ be any Mcshane partitions of $[0, 1]$. Then

$$f(\mathcal{P}') = \sum_{i=1}^{k} f(t'_i)(b'_i - a'_i)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{h} f(t'_i)\mu([a'_i, b'_i] \cap [a''_j, b''_j])$$

and

$$f(\mathcal{P}'') = \sum_{j=1}^{h} f(t''_j)(b''_j - a''_j)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{h} f(t''_j)\mu([a'_i, b'_i] \cap [a''_j, b''_j])$$

Hence $\|f(\mathcal{P}') - f(\mathcal{P}'')\| = \|\sum_{i=1}^{k} \sum_{j=1}^{h} f(t'_i)\mu([a'_i, b'_i] \cap [a''_j, b''_j])\|$. Therefore by Theorem 2.2, $f : [0, 1] \rightarrow X$ is Mcshane integrable if and only if for each $\varepsilon > 0$ there exists a gauge $\delta$ on $[0, 1]$ such that $\|\sum_{i=1}^{k} \sum_{j=1}^{h} f(t'_i)\mu([a'_i, b'_i] \cap [a''_j, b''_j])\| < \varepsilon$ whenever $\mathcal{P}' = \{(a'_i, b'_i], t'_i) : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{(a''_j, b''_j), t''_j) : 1 \leq j \leq h\}$ are Mcshane partitions of $[0, 1]$ subordinate to $\delta$. \hfill $\Box$

The following is a main result of this paper.

Theorem 3.2. Let $X$ be a totally ordered AL-space. Then $f : [0, 1] \rightarrow X$ is Mcshane integrable if and only if $f$ is absolutely Mcshane integrable.

Proof. Suppose that $f : [0, 1] \rightarrow X$ is absolutely Mcshane integrable. Let $\varepsilon > 0$ be given. Then there exists a gauge $\delta$ on $[0, 1]$ such that $\sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t'_i)\mu([a'_i, b'_i] \cap [a''_j, b''_j])\| < \varepsilon$ whenever $\mathcal{P}' = \{(a'_i, b'_i], t'_i) : 1 \leq i \leq k\}$ and $\mathcal{P}'' = \{(a''_j, b''_j), t''_j) : 1 \leq j \leq h\}$ are Mcshane partitions of $[0, 1]$ subordinate to $\delta$. Therefore $\|\sum_{i=1}^{k} \sum_{j=1}^{h} f(t'_i)\mu([a'_i, b'_i] \cap [a''_j, b''_j])\| < \varepsilon$ whenever
\[ \mathcal{P}' = \{([a'_i, b'_i], t'_i); 1 \leq i \leq k\} \] and \[ \mathcal{P}'' = \{([a''_j, b''_j], t''_j); 1 \leq j \leq h\} \] are McShane partitions of \([0, 1]\] subordinate to \(\delta\). By Lemma 3.1, \(f\) is McShane integrable.

For the converse, suppose that \(f : [0, 1] \rightarrow X\) is McShane integrable. Let \(\varepsilon > 0\) be given. Then there exists a gauge \(\delta\) on \([0, 1]\) such that

\[\|\sum_{i=1}^{k} f(t_i)(b_i - a_i) - \int_0^1 f \, d\mu\| < \frac{\varepsilon}{2}\]

whenever \(\mathcal{P} = \{([a_i, b_i], t_i); 1 \leq i \leq k\}\) is a McShane partition of \([0, 1]\) subordinate to \(\delta\).

Let \(\mathcal{P}' = \{([a'_i, b'_i], t'_i); 1 \leq i \leq k\}\) and \(\mathcal{P}'' = \{([a''_j, b''_j], t''_j); 1 \leq j \leq h\}\) be any McShane partitions of \([0, 1]\) subordinate to \(\delta\). Define \(t'_{ij} = t'_i\) and \(t''_{ij} = t''_j\) if \(f(t'_i) \geq f(t''_j)\) and define \(t'_{ij} = t'_j\) and \(t''_{ij} = t''_i\) if \(f(t'_i) < f(t''_j)\). Then \(f(t'_{ij}) - f(t''_{ij}) \in X_+\) and

\[\|f(t'_{ij}) - f(t''_{ij})\| = \|f(t'_i) - f(t''_j)\|\]

for \(1 \leq i \leq k, 1 \leq j \leq h\). Moreover \(\{([a'_i, b'_i] \cap [a''_j, b''_j], t'_{ij}); 1 \leq i \leq k, 1 \leq j \leq h\}\) and \(\{([a'_i, b'_i] \cap [a''_j, b''_j], t''_{ij}); 1 \leq i \leq k, 1 \leq j \leq h\}\) are both McShane partitions of \([0, 1]\) subordinate to \(\delta\). Hence

\[\|\sum_{i=1}^{k} \sum_{j=1}^{h} f(t'_{ij})\mu([a'_i, b'_i] \cap [a''_j, b''_j]) - \int_0^1 f \, d\mu\| < \frac{\varepsilon}{2}\]

and

\[\|\sum_{i=1}^{k} \sum_{j=1}^{h} f(t''_{ij})\mu([a'_i, b'_i] \cap [a''_j, b''_j]) - \int_0^1 f \, d\mu\| < \frac{\varepsilon}{2}\]

Therefore

\[\|\sum_{i=1}^{k} \sum_{j=1}^{h} [f(t'_{ij}) - f(t''_{ij})]\mu([a'_i, b'_i] \cap [a''_j, b''_j])\| < \varepsilon.\]

Since \(X\) is an AL-space and \(f(t'_{ij}) - f(t''_{ij}) \in X_+\) for \(1 \leq i \leq k, 1 \leq j \leq h,\)

\[\sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t'_{ij}) - f(t''_{ij})\|\mu([a'_i, b'_i] \cap [a''_j, b''_j])\]

\[= \sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t'_{ij}) - f(t''_{ij})\|\mu([a'_i, b'_i] \cap [a''_j, b''_j])\]

\[= \|\sum_{i=1}^{k} \sum_{j=1}^{h} [f(t'_{ij}) - f(t''_{ij})]\mu([a'_i, b'_i] \cap [a''_j, b''_j])\| < \varepsilon.\]
Therefore $f: [0, 1] \rightarrow X$ is absolutely Mcshane integrable. □ □

**Corollary 3.3.** Let $X$ be a totally ordered AL-space and $Y$ a Banach space. If $f: [0, 1] \rightarrow X$ is Mcshane integrable and $g: X \rightarrow Y$ is Lipschitz continuous, then the composite function $g \circ f: [0, 1] \rightarrow Y$ is Mcshane integrable.

**Proof.** Let $\varepsilon > 0$ be given. Since $g$ is Lipschitz continuous, there exists a $K > 0$ such that $\|g(x') - g(x)\| \leq K\|x' - x\|$ for all $x', x \in X$. Since $f$ is Mcshane integrable, by Theorem 3.2, $f$ is absolutely Mcshane integrable. Hence there exists a gauge $\delta$ on $[0, 1]$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{h} \|f(t_i') - f(t_j'')\| \mu([a_i', b_i'] \cap [a_j'', b_j'']) < \frac{\varepsilon}{K}$$

whenever $P' = \{([a_i', b_i'], t_i') : 1 \leq i \leq k\}$ and $P'' = \{([a_j'', b_j''], t_j'') : 1 \leq j \leq h\}$ are Mcshane partitions of $[0, 1]$ subordinate to $\delta$.

Hence

$$\| \sum_{i=1}^{k} \sum_{j=1}^{h} [(g \circ f)(t_i') - (g \circ f)(t_j'')] \mu([a_i', b_i'] \cap [a_j'', b_j'']) \|$$

$$\leq \sum_{i=1}^{k} \sum_{j=1}^{h} \| (g \circ f)(t_i') - (g \circ f)(t_j'') \| \mu([a_i', b_i'] \cap [a_j'', b_j''])$$

$$\leq K \sum_{i=1}^{k} \sum_{j=1}^{h} \| f(t_i') - f(t_j'') \| \mu([a_i', b_i'] \cap [a_j'', b_j'']) < \varepsilon$$

whenever $P' = \{([a_i', b_i'], t_i') : 1 \leq i \leq k\}$ and $P'' = \{([a_j'', b_j''], t_j'') : 1 \leq j \leq h\}$ are Mcshane partitions of $[0, 1]$ subordinate to $\delta$. Thus $g \circ f$ is Mcshane integrable by Lemma 3.1. □ □

**References**


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