Bull. Korean Math. Soc. **34**(1997), No. 1, pp. 19–28

OPERATORS ON A FINITE DIMENSIONAL SPACE

EUNGIL KO

1. Introduction

Let **H** and **K** be separable, complex Hilbert spaces and $\mathcal{L}(\mathbf{H},\mathbf{K})$ denote the space of all linear, bounded operators from **H** to **K**. If **H**=**K**, we write $\mathcal{L}(\mathbf{H})$ in place of $\mathcal{L}(\mathbf{H},\mathbf{K})$. An operator T in $\mathcal{L}(\mathbf{H})$ is called hyponormal if $TT^* \leq T^*T$, or equivalently, if $\|T^*h\| \leq \|Th\|$ for each h in **H**. In [Pu], M. Putinar constructed a universal functional model for hyponormal operators. A linear bounded operator S on **H** is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(\mathbf{H})$$

such that $\Phi(z) = S$, where as usual z stands for the identity function on **C**, and $C_0^m(\mathbf{C})$ stands for the space of compactly supported functions on **C**, continuously differentiable of order m, $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to a closed invariant subspace. M. Putinar in [Pu] showed that every hyponormal operator is subscalar of order 2. In this paper we show that every operator on a finite dimensional complex space is subscalar. The techniques which are developed in this paper will be useful to characterize several classes of operators on an infinite dimensional Hilbert space. This paper has four sections. In section two we have included the preliminaryfacts. Section three deal with Putinar theorem. In section four we show the main theorems.

Received December 20, 1995. Revised December 9, 1996.
1991 AMS Subject Classification: 47B20, 47B38.
Key words and phrases: Subscalar operators, Putinar's theorem, Finite rank operators.
This paper was supported by Ewha Research Grant.

2. Preliminaries

Let z be the coordinate in the complex plane $\mathbf{C}$ and let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space $\mathbf{H}$ and a bounded (connected) open subset U of $\mathbf{C}$. We shall denote by $L^2(U, \mathbf{H})$ the Hilbert space of measurable functions $f : U \to \mathbf{H}$, such that

$$\|f\|_{2,U} = \left(\int_U \|f(z)\|^2 d\mu(z)\right)^{\frac{1}{2}} < \infty$$

The space of functions $f \in L^2(U, \mathbf{H})$ which are analytic functions in U (i.e., $\bar{\partial} f = 0$) is denoted by $A^2(U, \mathbf{H})$. $A^2(U, \mathbf{H})$ is called the Bergman space for U. Note that $A^2(U, \mathbf{H})$ has a natural inner product and norm from $L^2(U, \mathbf{H})$.

The Bergman operator for the open set U is the operator S defined on $A^2(U, \mathbf{H})$ by $(Sf)(z) = zf(z)$. Since U is bounded, S is a bounded operator on a Hilbert space $A^2(U, \mathbf{H})$.

The proof is elementary but some of the facts established will be needed later. Throughout this paper we will assume that $0 \in D$, where D is a bounded open set.

PROPOSITION 2.1. ([Co]) *If $f \in A^2(U, \mathbf{H})$, $a \in U$, and dist$(a, \partial U) > r > 0$, then*

$$\|f(a)\| \leq \frac{1}{r\sqrt{\pi}} \|f\|_{2,U}.$$

PROPOSITION 2.2. *If S is the Bergman operator for the bounded open set D, then S is bounded below.*

Proof. This immediately follows from [Co, Corollary 10.7, page 177]. □

Let us define now a special Sobolev type space. Let U be again a bounded open subset of $\mathbf{C}$ and m be a fixed non-negative integer. The vecter valued Sobolev space $W^m(U, \mathbf{H})$ with respect to $\bar{\partial}$ and of order m will be the space of those functions f in $L^2(U, \mathbf{H})$ whose derivatives $\bar{\partial} f, \ldots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, \mathbf{H})$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U}$$

$W^m(U, \mathbf{H})$ becomes a Hilbert space contained continuously in $L^2(U, \mathbf{H})$.

We next discuss the fact concerning the multiplication operator by z on $W^m(U, \mathbf{H})$. Let U be a (connected) bounded open subset of $\mathbf{C}$ and let m be a non-negative integer. The linear operator M of multiplication by z on $W^m(U, \mathbf{H})$ is continuous and it has a spectral distribution of order m, defined by the relation

$$\Phi_M : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(W^m(U, \mathbf{H})), \quad \Phi_M(f) = M_f$$

Therefore, M is a scalar operator of order m.

3. Putinar Theorem

Let $T \in \mathcal{L}(\mathbf{H})$. Then for a given open bounded subset D of $\mathbf{C}$, $z - T$ acts (linearly and) continuously on the space $W^2(D, \mathbf{H})$.

LEMMA 3.1. ([Pu], Lemma 1.1) *If U and V are bounded connected open sets in $\mathbf{C}$, and if V is relatively compact in U, then there is a constant $c > 0$, such that*

$$\|f\|_{\infty,V} \leq c\|f\|_{2,U}$$

for every f in $A^2(U, \mathbf{H})$.

PROPOSITION 3.2. ([Pu], Proposition 2.1) *For a bounded disk D in the complex plane there is a constant C_D, such that for an arbitrary operator T in $\mathcal{L}(\mathbf{H})$ and f in $W^2(D, \mathbf{H})$ we have*

$$\|(I - P)f\|_{2,D} \leq C_D(\|(z - T)^*\bar{\partial} f\|_{2,D} + \|(z - T)^*\bar{\partial}^2 f\|_{2,D})$$

where P denotes the orthogonal projection of $L^2(D, \mathbf{H})$ onto the Bergman space $A^2(D, \mathbf{H})$.

COROLLARY 3.3. ([Pu], Corollary 2.2) *If T is hyponormal, then*

$$\|(I - P)f\|_{2,D} \leq C_D(\|(z - T)\bar{\partial} f\|_{2,D} + \|(z - T)\bar{\partial}^2 f\|_{2,D})$$

THEOREM 3.4. ([Pu], Theorem 1) *Any hyponormal operator is subscalar of order 2.*

Proof. Let T be a hyponormal operator on the Hilbert space $\mathbf{H}$. Consider an arbitrary bounded open subset D of $\mathbf{C}$ and the quotient space

$$H(D) = \frac{W^2(D, \mathbf{H})}{cl(z - T)W^2(D, \mathbf{H})}$$

endowed with the Hilbert space norm. The class of a vector f or an operator A on this quotient will be denoted by $\tilde{f}$, respectively $\tilde{A}$.

Note that M, the operator of multiplication by z on $W^2(D, \mathbf{H})$, leaves invariant ran $(z - T)$, hence $\tilde{M}$ is well defined.

On the other hand, the map

$$\Phi : C_0^2(\mathbf{C}) \longrightarrow \mathcal{L}(W^2(D, \mathbf{H})), \quad \Phi(f) = M_f$$

is a spectral distribution for M, order 2. Thus the operator M is C^2-scalar. Since $\overline{ran(z - T)}$ is invariant under every operator M_f, $f \in C_0^2(\mathbf{C})$, we infer that $\tilde{M}$ is a C^2-scalar operator with spectral distribution $\tilde{\Phi}$.

Define

$$V : \mathbf{H} \longrightarrow \frac{W^2(D, \mathbf{H})}{cl(z - T)W^2(D, \mathbf{H})}$$

by $V(h) = \widetilde{1 \otimes h}$ where $1 \otimes h$ denotes the constant function h. Then

$$VT = \tilde{M}V.$$

Indeed, $VTh = (1 \otimes Th)\tilde{} = \widetilde{z \otimes h} = \tilde{M}(\widetilde{1 \otimes h}) = \tilde{M}Vh$. In particular ran V is an invariant subspace for $\tilde{M}$. In order to conclude the proof of this theorem, it is enough to show the following lemma. □

LEMMA 3.5. ([Pu], Lemma 2.3) *Let D be a bounded disk which contains $\sigma(T)$. Then the operator V is one-to-one and has closed range.*

Proof. We have to prove the following assertion; if h_n in $\mathbf{H}$ and f_n in $W^2(D, \mathbf{H})$ are sequences such that

$$\lim_{n \to \infty} \|(z - T)f_n + h_n\|_{W^2} = 0 \tag{1}$$

then $\lim_{n \to \infty} h_n = 0$. The assumption (1) implies

$$\lim_{n \to \infty} (\|(z - T)\bar{\partial} f_n\|_{2,D} + \|(z - T)\bar{\partial}^2 f_n\|_{2,D}) = 0.$$

By Corollary 3.3,

$$\lim_{n \to \infty} \|(I - P)f_n\|_{2,D} = 0.$$

Then by (1),

$$\lim_{n \to \infty} \|(z - T)Pf_n + h_n\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n \to \infty} \|Pf_n(z) + (z - T)^{-1}h_n\| = 0$$

uniformly by the preceding consequence of Proposition 3.2.
Hence,

$$\|\frac{1}{2\pi i}\int_{\Gamma} Pf_n(z)dz + h_n\| \longrightarrow 0.$$

But $\int_\Gamma Pf_n dz = 0$. Hence, $\lim_{n\to\infty} h_n = 0$ □

4. Main theorems

In this section we finally prove the central theorem of the paper, which is the following.

THEOREM 4.1. *Let* $\mathbf{H}$ *be a finite dimensional, complex space and let* $A \in \mathcal{L}(\mathbf{H})$. *Then* A *is a subscalar operator.*

The proof of this theorem will be accomplished by making some preliminary reductions and then proving a sequence of lemmas. The Jordan structure theorem says that every square matrix A over the complex numbers $\mathbf{C}$ is similar to another matrix B which is a direct sum of Jordan cells. That is, B can be written in the block form

$$\bigoplus_{n=1}^{k} B_n = B$$

and each B_n has the form

$$\begin{pmatrix} \alpha_n & 1 & 0 & \cdots & \\ 0 & \alpha_n & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \alpha_n & 1 \\ 0 & \cdots & \cdots & 0 & \alpha_n \end{pmatrix}$$

for some α_n in $\mathbf{C}$. The numbers $\{\alpha_1, \cdots, \alpha_k\}$ can be identified as the spectrum or set of eigenvalues of A.

By similarity, A is subscalar if and only if B is subscalar. Therefore, it is enough to show that B is subscalar. The following is easy to prove.

PROPOSITION 4.2. *Let* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *be two Hilbert spaces. If* $B_i \in \mathcal{L}(\mathbf{H}_i)$ *are subscalar, then* $\oplus_{i=1}^{2} B_i$ *is subscalar.*

From Proposition 4.2, B is subscalar provided B_n are for any n. To show Theorem 4.1, it suffices to show that B_n with $\alpha_n = 0$ is subscalar by translation. We have reduced the Theorem 4.1 to the Theorem 4.3.

THEOREM 4.3. *Let T be a matrix on n-dimensional, complex space such that*

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Then T is a subscalar operator of order $2n$.

LEMMA 4.4. *If $g \in W^{2m}(D, \mathbf{H})$ is such that $\| - zg\|_{W^{2m}} < \delta$, then for $i = 0, 1, \cdots, 2m-2$, there exist $h_i \in A^2(D, \mathbf{H})$ such that*

$$\|(g, \bar{\partial} g, \cdots, \bar{\partial}^{2m-2} g) - (h_0, h_1, \cdots, h_{2m-2})\|_{2,D} < 2C_D\delta$$

where C_D is a constant depending on D.

Proof. Let $\oplus_m \mathbf{H}$ denote the direct sum of m copies of $\mathbf{H}$. Let P denote the orthogonal projection of $\oplus_{2m-1} L^2(D, \mathbf{H})$ onto the Bergman space $\oplus_{2m-1} A^2(D, \mathbf{H})$. By Proposition 3.2 with $\oplus_n T = (0)$,

$$\begin{aligned} \|(I-P)(g, \bar{\partial} g, \cdots, \bar{\partial}^{2m-2} g)\|_{2,D} &\leq C_D(\| - z(\bar{\partial} g, \cdots, \bar{\partial}^{2m-1} g)\|_{2,D} \\ &\quad + \| - z(\bar{\partial}^2 g, \cdots, \bar{\partial}^{2m} g)\|_{2,D}) \\ &< 2C_D\delta. \end{aligned}$$

Set $P(g, \bar{\partial} g, \cdots, \bar{\partial}^{2m-2} g) = (h_0, h_1, \cdots, h_{2m-2})$. □

THEOREM 4.5. *g_0 is not in $\overline{(T-z)W^{2n}(D, \mathbf{H})}$ where $g_0(z) = e_0$.*

Proof. We want to show that g_0 does not belong to $\overline{(T-z)W^{2n}(D, \mathbf{H})}$. If not, there exists $f = (f_1, f_2, \cdots, f_n) \in W^{2n}(D, \mathbf{H})$ such that

$$\|(T-z)f - g_0\|_{W^{2n}} < \epsilon$$

Note that

$$(T-z)f(z) = \begin{pmatrix} -zf_1(z) + f_2(z) \\ \vdots \\ -zf_{n-1}(z) + f_n(z) \\ -zf_n(z) \end{pmatrix}$$

That implies

$$\begin{cases} \|-zf_1+f_2-1\|_{2,D}^2+\sum_{i=1}^{2n}\|-z\bar{\partial}^i f_1+\bar{\partial}^i f_2\|_{2,D}^2 & < \epsilon^2 \quad (1,1) \\ \sum_{i=0}^{2n}\|-z\bar{\partial}^i f_2+\bar{\partial}^i f_3\|_{2,D}^2 & < \epsilon^2 \quad (1,2) \\ \vdots & \vdots \\ \sum_{i=0}^{2n}\|-z\bar{\partial}^i f_n\|_{2,D}^2 & < \epsilon^2 \quad (1,n) \end{cases}$$

Claim: For $i = 1, 2, \cdots\cdots, n$,

$$\|-z(\bar{\partial}f_i, \cdots, \bar{\partial}^{2i}f_i)\|_{2,D} < (1 + R + \cdots + R^{n-i})\epsilon$$

where $R \equiv 2C_D + (2\tilde{C_D}C_D + 1)/c$, C_D is a constant depending on D, $\tilde{C_D} = sup_{z\in D}|z|$, and c is a Bergman constant from Proposition 2.2.

Now we verify the above claim. If $i = n$, it is clear from (1,n) assuming of course that $\epsilon < 1$. Using the induction, assume that it is true for $i = t$. We want to show that

$$\|-z(\bar{\partial}f_{t-1}, \cdots, \bar{\partial}^{2(t-1)}f_{t-1})\|_{2,D} < (1 + R + \cdots + R^{n-(t-1)})\epsilon$$

Set $\gamma_t \equiv 1 + R + \cdots + R^{n-t}$.
By the induction assumption,

$$\begin{cases} \|-z(\bar{\partial}f_t, \cdots, \bar{\partial}^{2t-1}f_t)\|_{2,D} < \gamma_t\epsilon & (2,1) \\ \|-z(\bar{\partial}^2 f_t, \cdots, \bar{\partial}^{2t}f_t)\|_{2,D} < \gamma_t\epsilon & (2,2) \end{cases}$$

By Lemma 4.4, for $(h_1, \cdots, h_{2t-2}) \in \oplus_{2t-1}A^2(D, \mathbf{H})$

$$\|(\bar{\partial}f_t, \cdots, \bar{\partial}^{2t-2}f_t) - (h_1, \cdots, h_{2t-2})\|_{2,D} < 2C_D\gamma_t\epsilon. \quad (3)$$

$$\|z(\bar{\partial}f_t, \cdots, \bar{\partial}^{2t-2}f_t) - z(h_1, \cdots, h_{2t-2})\|_{2,D} < 2\tilde{C_D}C_D\gamma_t\epsilon. \quad (3)'$$

By $(3)'$ and (2,1),

$$\|z(h_1, \cdots, h_{2t-2})\|_{2,D} < (2\tilde{C_D}C_D + 1)\gamma_t\epsilon.$$

By Proposition 2.2, there exists $c > 0$ such that

$$c\|h_i\|_{2,D} \le \|zh_i\|_{2,D}.$$

Therefore,

$$\|(h_1, \cdots, h_{2t-2})\|_{2,D} < \frac{2\tilde{C_D}C_D + 1}{c}\gamma_t\epsilon. \quad (4)$$

By (3) and (4),

$$\begin{aligned} \|(\bar\partial f_t, \cdots, \bar\partial^{2t-2} f_t)\|_{2,D} &< 2C_D\gamma_t\epsilon + \frac{2\tilde{C_D}C_D+1}{c}\gamma_t\epsilon \\ &= R\gamma_t\epsilon. \end{aligned} \tag{5}$$

Using (5) and (1,t-1),

$$\begin{aligned} \|-z(\bar\partial f_{t-1}, \cdots, \bar\partial^{2(t-1)} f_{t-1})\|_{2,D} &< (1+R\gamma_t)\epsilon \\ &= (1+R+\cdots+R^{n-(t-1)})\epsilon \\ &= \gamma_{t-1}\epsilon. \end{aligned}$$

The claim has been proved. Now we want to complete the proof of Theorem 4.5. By the claim,

$$\begin{cases} \|-z(\bar\partial f_1, \bar\partial^2 f_1)\|_{2,D} < \gamma_1\epsilon & (6,1) \\ \|-z(\bar\partial f_2, \bar\partial^2 f_2, \bar\partial^3 f_2, \bar\partial^4 f_2)\|_{2,D} < \gamma_2\epsilon. & (6,2) \\ \quad\vdots & \\ \|-z(\bar\partial f_n, \cdots, \bar\partial^{2n} f_n)\|_{2,D} < \gamma_n\epsilon. & (6,n) \end{cases}$$

By (6,1) and Proposition 2.2, for $t = 0, 1, \cdots\cdots, n$

$$\|(I-P)f_t\|_{2,D} < 2C_D\gamma_t\epsilon. \tag{7}$$

where P denotes the orthogonal projection of $L^2(D)$ onto the Bergman space $A^2(D)$. Set $\hat{h} \equiv (Pf_1, Pf_2, 0, \cdots\cdots, Pf_n)^t$. Then by (7) ,

$$\begin{aligned} \|f-\hat{h}\|_{2,D} &\le \|f_1 - Pf_1\|_{2,D} + \cdots\cdots + \|f_n - Pf_n\|_{2,D} \\ &< 2C_D(\gamma_1+\gamma_2+\cdots+\gamma_n)\epsilon. \end{aligned}$$

Then, for $\sigma(T) \subset B(0,r) \subset \overline{B(0,r)} \subset D$, we know that $\|(T-z)f - e_0\|_{2,D} < \epsilon$ since $\|(T-z)f - e_0\|_{W^{2n}} < \epsilon$ and

$$\begin{aligned} \|(T-z)\hat{h} - e_0\|_{2,D} &\le \|(T-z)\hat{h} - (T-z)f\|_{2,D} \\ &\quad + \|(T-z)f - e_0\|_{2,D} \\ &< (sup_{z\in D}\|T-z\|)\|\hat{h} - f\|_{2,D} + \epsilon \\ &< 2C_D F(\gamma_1+\cdots+\gamma_n)\epsilon + \epsilon \end{aligned}$$

where $F = sup_{z\in D}\|T-z\|$. By Lemma 3.1, there exists a constant $d > 0$ such that

$$\begin{aligned}\|(T-z)\hat{h}-e_0\|_{\infty,B(0,r)} &\leq d\|(T-z)\hat{h}-e_0\|_{2,D}\\ &< 2dC_DF(\gamma_1+\gamma_2+\cdots+\gamma_n)\epsilon+d\epsilon\end{aligned}$$

Therefore,

$$\begin{aligned}1=\|e_0\| &= \|\frac{1}{2\pi i}\int_{\partial B(0,r)}(T-z)^{-1}e_0dz\|\\ &= \|\frac{1}{2\pi i}\int_{\partial B(0,r)}((T-z)^{-1}e_0-\hat{h}(z))dz\|\\ &= \|\frac{1}{2\pi i}\int_{\partial B(0,r)}(T-z)^{-1}(e_0-(T-z)\hat{h}(z))dz\|\\ &\leq \|(T-z)\hat{h}-e_0\|_{\infty,B(0,r)}\frac{1}{2\pi}\int_{\partial B(0,r)}\|(T-z)^{-1}\|dz\\ &< 2dEFC_D(\gamma_1+\gamma_2+\cdots+\gamma_n)\epsilon+dE\epsilon\end{aligned}$$

where $E=\frac{1}{2\pi}\int_{\partial B(0,r)}\|(T-z)^{-1}\|dz$. Since ϵ was arbitrary, we have a contradiction. Thus g_0 is not in $\overline{(T-z)W^{2n}(D,\mathbf{H})}$. □

Proof of Theorem 4.3: Let D be a bounded disk which cotains $\sigma(T)$. Consider the quotient space

$$H(D)=\frac{W^{2n}(D,\mathbf{H})}{cl(T-z)W^{2n}(D,\mathbf{H})}$$

where cl denotes the norm closure.

Let M_z be a multiplication operator with z on $W^{2n}(D,\mathbf{H})$. Then M_z is a C^{2n}-scalar subnormal operator and its spectral distribution is

$$\Phi: C_0^{2n}(\mathbf{C})\longrightarrow \mathcal{L}(W^{2n}(D,\mathbf{H})),\quad \Phi(f)=M_f$$

Since $\overline{ran(T-z)}$ is invariant under M_z, $\widetilde{M_z}$ is still a scalar operator of order $2n$, with $\tilde{\Phi}$ as spectral distribution.

Let V be the operator $V(h)=\widetilde{1\otimes h}$, from $\mathbf{H}$ into $H(D)$, denoting by $1\otimes h$ the constant function h. Then $VT=\widetilde{M_z}V$. In particular the range of V is an invariant subspace for $\widetilde{M_z}$. Thus the proof is completed by the following lemma.

LEMMA 4.6. *The operator V is one-to-one and has close range.*

Proof. Since V is the operator on n-dimensional space, ran V is closed. Assume that $Vh = 0$ and $h \neq 0$ where $h = \sum_{i=0}^{n-1} h_i e_i$. Since $h \neq 0$, there exist at least one i such that $h_i \neq 0$. Let $j (0 \leq j \leq n-1)$ be the largest index among such i's. Since ker $V \in$ Lat T, ker $V \in$ Lat T^{j-1}. Therefore,

$$T^{j-1}h = \begin{pmatrix} h_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

But, since $T^{j-1}h \in$ ker V, by Theorem 4.5, we have a contradiction. Thus ker $V = \{0\}$. □

Lemma 4.6 concludes the proof of Theorem 4.3, because the range of V is a closed invariant subspace for the scalar operator $\widetilde{M_z}$. □

LEMMA 4.7. ([RR], Proposition 06) *If A is a finite rank operator, then A is unitarily equivalent to an operator of the form $B \oplus 0$, where B is an operator on a finite dimensional space.*

COROLLARY 4.8. *If A is a finite rank operator, then A is a subscalar operator.*

ACKNOWLEDGEMENTS. The author wishes to thank Professor Scott W. Brown for his considerable guidance. Special thanks go to the referee for a careful reading of the original draft.

References

[Co] J. B. Conway, *Subnormal operators.* Pitman, London, 1981.

[Ko] Eungil Ko, *Subscalar and quasisubscalar operators.* Ph.D. thesis, Indiana University, 1993.

[Pu] M. Putinar, *Hyponormal operators are subscalar.* J. Operator Theory, **12**, 385-395, 1984.

[RR] H. Radjavi and P. Rosenthal, *Invariant subspaces.* Springer-Verlag, Berlin-Heidelberg-New York, 1973.

DEPARTMENT OF MATHEMATICS, EWHA WOMEN'S UNIVERSITY, SEOUL 120-750, KOREA