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ON THE COHEN-MACAULAYNESS OF THE ASSOCIATED GRADED RING OF AN EQUIMULTIPLE IDEAL

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1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity. By a local ring (R, m), we mean a Noetherian ring R which has a unique maximal ideal m. Let I be an ideal in a ring R and t an indeterminate over R. Then the Rees algebra $R[It]$ and the associated graded ring $gr_I(R)$ of I are defined to be

$$R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots$$

and

$$gr_I(R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

With these notations, we have a natural isomorphism

$$\frac{R[It]}{IR[It]} \cong gr_I(R).$$

U. Grothe, M. Herrmann and U. Orbanz in [2, Proposition 2.9] proved that the number of homogeneous elements among a system of parameters of $R[It]$ is bounded above by $dim(R) - l(I) + 2$. In this paper, we shall show that the number of homogeneous parameters for $gr_I(R)$ is bounded above by $2dim(R) - l(I) - dim(R/I)$. And we shall extend to equimultiple ideals results of J. Sally [7, Theorem 2], Valabrega-Valla [10, Proposition 3.1].

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2. Preliminaries

In this section we state several definitions and notations which are needed in our subsequent considerations. Let R be a Noetherian ring and I an ideal of R. Given an element $a \in R$, we define

$$v_I(a) = \begin{cases} n & if \quad a \in I^n \backslash I^{n+1}, \\ \infty & if \quad a \in \cap_{n \geq 1} I^n. \end{cases}$$

When $v_I(a) = n \neq \infty$, the residue class of a in I^n/I^{n+1} is called the leading form of a and denoted by a^* with $deg_I(a) = v_I(a)$. If $v_I(a) = \infty$, then we set $a^* = 0$.

LEMMA 2.1 ([10, COROLLARY 2.7]). *If I and $J = (b_1, \cdots, b_k)$ are ideals of a local ring (R, m), then $b_1^*, \cdots, b_k^*$ form a $gr_I(R)$-sequence if and only if $b_1, \cdots, b_k$ form an R-sequence and moreover for all $i, 1 \leq i \leq k$, and for all $n \geq 1$,*

$$(b_1, \cdots, b_i) \cap I^n = \sum_{j=1}^{i} I^{n - v_I(b_j)} b_j.$$

DEFINITION 2.2. Let (R, m) be a local ring and I an ideal of R. An ideal J contained in I is called a reduction of I if $JI^n = I^{n+1}$ for some integer $n \geq 0$. A reduction J of I is called a minimal reduction of I if J is minimal with respect to being a reduction of I. The reduction number of I is defined by $r(I) = \{n \geq 0 |$ there exists a minimal reduction J of I such that $JI^n = I^{n+1}\}$.

DEFINITION 2.3. Let (R, m) be a local ring and I be an ideal of R. Define $l(I)$ to be the Krull dimension of the graded ring

$$R[It]/mR[It] = R/m \oplus I/mI \oplus I^2/mI^3 \oplus \cdots$$

$l(I)$ is called the analytic spread of I. The ideal I is said to be equimultiple if $ht(I) = l(I)$.

The following result is known (c.f. Lemma 4.4 in [6] and Corollary in [1]). We will give an alternative proof.

PROPOSITION 2.4. *Let (R, m) be a local ring and I be an ideal of R. Then $ht(I) \leq l(I) \leq dim(R)$.*

Proof. By the definition of $l(I)$, we know that $l(I) = dim(gr_I(R) / mgr_I(R))$. Since $dim(gr_I(R)) = dim(R)$ (by Theorem 15.7 in [5]), we get that $l(I) \leq dim(R)$. To see the first inequalty, by Theorem 15.1 in [5], we have that $dim(gr_I(R)) - dim(R/I) \leq l(I)$, and hence $ht(I) \leq dim(R) - dim(R/I) \leq l(I)$. ∎

REMARKS 2.5. Let (R, m) be a local ring.
(a) Since $ht(I) \leq l(I) \leq dim(R)$, any m-primary ideal is equimultiple.
(b) If R/m is an infinite field, then $l(I)$ is the least number of elements generating a reduction of I ([6, Corollary of Theorem 2, pp 151]).

3. The Number of Homogeneous Parameters for $gr_I(R)$

DEFINITION 3.1. Let A be a d-dimensional Noetherian ring. A set $\{a_1, \cdots, a_d\} \subset A$ will be called a system of parameters for A if $\sqrt{(a_1, \cdots, a_d)A}$ is an intersection of maximal ideals m_i of A such that $dim(A_{m_i}) = d$. If in addition A is graded and $a_1, \cdots, a_d$ are homogeneous, then $\{a_1, \cdots, a_d\}$ will be called a homogeneous system of parameters.

LEMMA 3.2. *Let A be a Noetherian ring of finite Krull dimension. Then if $a_1, \cdots, a_s \in A$ is part of a system of parameters of A we have*

$$dim(A/(a_1, \cdots, a_s)A) = dim(A) - s.$$

Proof. See Theorem 13.6 in [5]. ∎

PROPOSITION 3.3. *Let (R, m) be a local ring and I be a proper ideal of R such that $ht(I) > 0$. Let N be the unique homogeneous maximal ideal of $gr_I(R)$, and $h_1, \cdots, h_r \in N$ be part of a system of parameters of $gr_I(R)$. Then the number of homogeneous elements among $\{h_1, \cdots, h_r\}$ is at most $2dim(R) - l(I) - dim(R/I)$.*

Proof. Let $G = gr_I(R)$ and $G_+ = \oplus_{n\geq 1} I^n/I^{n+1}$. Without loss of generality we may assume that $h_1, \cdots, h_u$ are homogeneous elements of degree 0, $h_{u+1}, \cdots, h_{u+v}$ are homogeneous elements of positive degree, and $h_{u+v+1}, \cdots, h_r$ are non-homogeneous elements, where $0 \leq u, v \leq r$. Then by Lemma 3.2, we see that

$$l(I) = dim(G/mG) \leq dim(G) - u$$

and

$$dim(R/I) = dim(G/G_+) \leq dim(G) - v.$$

This gives that $l(I) + dim(R/I) \leq 2dim(G) - (u + v)$. Since $dim(G) = dim(R)$ ([5, Thedrem 15.7]) we see that $u + v \leq 2dim(R) - l(I) - dim(R/I)$. ∎

COROLLARY 3.4. *Lrt (R, m) be a local ring and let I be an ideal of R such that $ht(I) > 0$. Then the following conditions are equivalent.*
(a) $gr_I(R)$ has a homogeneous system of parameters.
(b) $dim(R) = dim(R/I) + l(I)$.

Proof. (a) $\Rightarrow$ (b). By Proposition 3.3 we see that $dim(gr_I(R)) \leq 2dim(R) - l(I) - dim(R/I)$. Since $dim(gr_I(R)) = dim(R)$ ([5, Theorem 15.7]) we have that $dim(R/I) + l(I) \leq dim(R)$. To see the other inequalty, by Theorem 15.1 in [5], we know that $dim(gr_I(R)) - dim(R/I) \leq l(I)$, and hence $dim(R) \leq dim(R/I) + l(I)$, which gives the assertion.
(b) $\Rightarrow$ (a). The following is a proof in [2, Proposition 2.6]. Let $t = dim(R)$ and $s = l(I)$. Let $b_1, \cdots, b_t$ be a system of parameters mod$I$. Then

$$dim(\frac{gr_I(R)}{(b_1^*, \cdots, b_t^*)gr_I(R)}) = l(I),$$

and $gr_I(R)/(b_1^*, \cdots, b_t^*)gr_I(R)$ has a homogeneous system of parameters $a_1', \cdots, a_s'$ ([9, (0.36)]). Let $a_1^*, \cdots, a_s^*$ in $gr_I(R)$ be any homogeneous inverse images of $a_1^{'}, \cdots, a_s^{'}$ respectively. Then

$$\begin{aligned} 0 &= dim(\frac{gr_I(R)/(b_1^*, \cdots, b_t^*)gr_I(R)}{(a_1^{'}, \cdots, a_s^{'})}) \\ &= dim(\frac{gr_I(R)}{(b_1^*, \cdots, b_t^*, a_1^*, \cdots, a_s^*)gr_I(R)}) \\ &\leq dim(gr_I(R)) - (t+s) \\ &= dim(R) - (t+s) \\ &= 0. \end{aligned}$$

So we have that $\{b_1^*, \cdots, b_t^*, a_1^*, \cdots, a_s^*\}$ is a homogeneous system of parameters of $gr_I(R)$. ∎

REMARKS 3.5. (1) ([2, Corollary 2.7]) The proof of (b) ⇒(a) above shows that, if $a_1, \cdots, a_s$ generate a minimal reduction of I, where $s = l(I)$, and $b_1, \cdots, b_t$ is a system of parameters $modI$, and $t + s = dim(R)$, then $\{b_1^*, \cdots, b_t^*, a_1^*, \cdots, a_s^*\}$ is a homogeneous system of parameters for $gr_I(R)$, where $b_1^*, \cdots, b_t^*$ are elements of degree 0, and $a_1^*, \cdots, a_s^*$ are elements of degree 1.
(2) If (R, m) is a local ring of $dim(R) = d$ and I is an m-primary ideal, then $gr_I(R)$ has a homogeneous system of parameters, i.e., $a_1^*, \cdots, a_d^*$ is a homogeneous system of parameters for $gr_I(R)$, where $a_1, \cdots, a_d$ generate a minimal reduction of I.
(3) ([9, (0.36)]) If $A = \oplus_{n \geq 0} A_n$ is a non-negatively graded ring and A_0 is an Artinian local ring, then the graded rimg A has a homogeneous system of parameters.

4. A Generalization of the Valabrega and Valla's Result

As an application of the Valabrega and Valla's result in [10], we know that if (R, m) is a d-dimensional Cohen-Macaulay local ring and I is an m-primary ideal satisfying $I^2 = (a_1, \cdots, a_d)I$ for some minimal reduction $(a_1, \cdots, a_d)$ of I, then $a_1^*, \cdots, a_d^*$ in I/I^2 form a $gr_I(R)$-sequence and hence $gr_I(R)$ is Cohen-Macaulay. As an extension of this result, we show in Theorem 4.1 the analogous result for an equimultiple ideal.

THEOREM 4.1. *Let (R, m) be a Cohen-Macaulay local ring and I be an equimultiple ideal satisfying $I^2 = (a_1, \cdots, a_s)I$ for some minimal reduction $a_1, \cdots, a_s$ of I, where $s = l(I)$. Assume that R/I is Cohen-Macaulay. Then $gr_I(R)$ is Cohen-Macaulay.*

Proof. By Remarks 3.5.(1), we know that $\{b_1^*, \cdots, b_t^*, a_1^*, \cdots, a_s^*\}$ is a homogeneous system of parameters for $gr_I(R)$, where $b_1, \cdots, b_t$ is a system of parameters $mod I$. Since R is Cohen-Macaulay, we have that $dim(R) = dim(R/I) + ht(I)$, and hence $t + s = dim(R) = gr_I(R)$ ([5, Theorem 15.7]). It suffices to show that $b_1^*, \cdots, b_t^*, a_1^*, \cdots, a_s^*$ is a $gr_I(R)$-sequence by [3]. To see this we have to consider equivalent conditions of Lemma 2.1. First, $b_1, \cdots, b_t, a_1, \cdots, a_s$ is a system of parameters for R, and hence it is an R-sequence since (R, m) is Cohen-Macaulay local ring. Secondly, we have to show that for all $n > 0$,

$$(b_1, \cdots, b_t, a_1, \cdots, a_s) \cap I^n = (b_1, \cdots, b_t)I^n + (a_1, \cdots, a_s)I^{n-1}.$$

For $n > 1$, we have that $(b_1, \cdots, b_t, a_1, \cdots, a_s) \cap I^n = I^n$ since $I^n \subseteq (a_1, \cdots, a_s)$, and $(b_1, \cdots, b_t)I^n + (a_1, \cdots, a_s)I^{n-1} = (b_1, \cdots, b_t)I^n + I^n = I^n$ since $I^2 = (a_1, \cdots, a_s)I$.
For $n = 1$, we have that

$$\begin{aligned}(b_1, \cdots, b_t, a_1, \cdots, a_s) \cap I &= ((b_1, \cdots, b_t) \cap I) + ((a_1, \cdots, a_s) \cap I) \\ &= (b_1, \cdots, b_t)I + (a_1, \cdots, a_s)\end{aligned}$$

since R/I is Cohen-Macaulay. This finishes the proof. ∎

The next example shows that Theorem 4.1 is false without some restriction on R/I.

EXAMPLE 4.2. Let $R = [[X, Y, Z]]$ and $I = (X^2, XYZ, Y^2)R$, where k is an infinite field and X, Y, Z are indeterminates. Then I is an equimultiple ideal since $(X^2, Y^2) \subseteq I = (X, Y)^2$ and $I^2 = (X^2, Y^2)I$. But R/I is not Cohen-Macaulay since $(I :_R XY) = (X, Y, Z)R$. Hence $R[It]$ is not Cohen-Macaulay by Theorem 3.1 in [4]. Therefore $gr_I(R)$ is not Cohen-Macaulay by Theorem 4.8 in [2].

COROLLARY 4.3 ([10, PROPOSITION 3.1]). *Let (R, m) be a d-dimensional Cohen-Macaulay local ring and I an m-primary ideal satisfying $I^2 = (a_1, \cdots, a_d)I$ for some minimal reduction $a_1, \cdots, a_d$ of I. Then $gr_I(R)$ is Cohen- Macaulay.*

The next example shows that Corollary 4.3 does not extend to the case where I has reduction number 2.

EXAMPLE 4.4. Let $R = [[t^3, t^4, t^5]]$ and $I = (t^3, t^4)R$, where k is an infinite field and t is an indeterminate. Then I is an m-primary ideal since $m^2 \subseteq I$, where $m = (t^3, t^4, t^5)R$, and $r(I) = 2$ since $I^3 = (t^3)I^2$. However, $t^5 \notin I$, but if $(t^5)^*$ is the image of t^5 in R/I then $(t^5)^*(I/I^2) = 0$ since $t^5 I \subseteq I^2$, and hence $depth(G_+) = 0$, where $G = gr_I(R)$. Hence $gr_I(R)$ is not Cohen-Macaulay.

COROLLARY 4.5([7, THEOREM 2]). *(R, m) be a d-dimensional Cohen-Macaulay local ring. Assume that there exist elements $x_1, \cdots, x_d$ in m such that $m^2 = (x_1, \cdots, x_d)m$. Then $gr_m(R)$ is Cohen-Macaulay.*

REMARK 4.6([8, THEOREM 2.1]). J. Sally showed that for any Cohen-Macaulay local ring (R, m), $gr_m(R)$ is Cohen-Macaulay if $r(m) \leq 2$.

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