Bull. Korean Math. Soc. **34**(1997), No. 1, pp. 93-102

CONVERGENCE OF APPROXIMATE SEQUENCES FOR COMPOSITIONS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

JONG SOO JUNG AND TAE HWA KIM

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E and let $T_1, \cdots T_N$ be nonexpansive mappings from C into itself (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x,\ y \in C$). We consider the fixed point problem for nonexpansive mappings : find a common fixed point, i.e., find a point in $\cap_{i=1}^N Fix(T_i)$, where $Fix(T_i) := \{x \in C : x = T_i x\}$ denotes the set of fixed points of T_i.

The most straightforward attempt to solve the fixed point problem for nonexpansive mappings is to iterate the mapping cyclically :

$$\begin{aligned} C \ni x_0 \longmapsto x_1 := T_1 x_0 \longmapsto \cdots \longmapsto x_N := T_N x_{N-1} \\ \longmapsto x_{N+1} := T_1 x_N \longmapsto \cdots \end{aligned} \tag{1}$$

For convenience, we set $T_n := T_{n \bmod N}$, where we let the mod N function take values in $\{1, \ \cdots, N\}$. Then we can rewirte (1) more compactly :

$$x_{n+1} := T_{n+1} x_n \quad \text{for all } n \ge 0 \text{ and } x_0 \in C. \tag{1}$$

Received November 13, 1996.

1991 AMS Subject Classification: 47H09, 49M05.

Key words and phrases: Nonexpansive mapping, fixed points, iteration, sunny and nonexpansive retraction.

The first author was supported by the Basic Science Research Institute Program, Ministry of Education, Korea, 1997, Project No. BSRI-97-1405, and the second author was also supported by Project No. BSRI-97-1440

Unfortunately, iteration scheme (1) can fail to produce a norm convergent sequence $\{x_n\}$ even if $N = 1$ and T_1 is firmly nonexpansive, since Genel and Lindenstrauss [4] presented an example where $\{x_n\}$ converges only weakly.

By approximating each nonexpansive mapping by Banach contractions, we can obtain the following modified version of (1): for a sequence λ_n in $[0, 1)$ converging to 0,

$$\begin{aligned} C \ni x_0 &\longmapsto x_1 := \lambda_1 a + (1 - \lambda_1) T_1 x_0 \longmapsto \cdots \\ &\longmapsto x_N := \lambda_N a + (1 - \lambda_N) T_N x_{N-1} \\ &\longmapsto x_{N+1} := \lambda_{N+1} a + (1 - \lambda_{N+1}) T_1 x_N \longmapsto \cdots \end{aligned}$$

or more compactly

$$(2) \quad x_{n+1} := \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n \quad \text{for all } n \geq 0 \text{ and } a,\ x_0 \in C.$$

In 1967, Halpern [8] suggested iteration scheme (2) for $N = 1$; see also Browder [2]. Ten years later, Lions [9] investigated the general case in Hilbert space under more restrictive condition on $\{\lambda_n\}$. In 1983, Reich [11] gave iteration scheme (2) for $N = 1$ in the case when E is uniformly smooth and $\lambda_n = n^{-a}$ with $0 < a < 1$. Since then, Wittmann [13] studied iteration scheme (2) for $N = 1$ in the case when E is a Hilbert space and $\{\lambda_n\}$ satisfies

$$0 \leq \lambda_n \leq 1, \ \lim_{n \to \infty} \lambda_n = 0, \ \sum_{n=0}^{\infty} \lambda_n = \infty \text{ and } \sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Shioji and Takahashi [12] improved Wittmann's result to Banach spaces under the assumption that each nonempty closed convex subset of C possesses the fixed point property for nonexpansive mappings.

Very recently, Bauschke [1] generalized Wittmann's result [13] to the case $N > 1$ in Hilbert space.

In this paper, we establish the strong convergence of $\{x_n\}$ defined by (2) in a unformly smooth Banach space with a weakly sequentially continuous duality mapping, which generalizes Bauschkė's result [1] to a Banach space setting. Our main result also improves Wittmann's result [13] for $N = 1$ to Banach spaces and partially generalizes a result by Lions [9].

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by (x, x^*). When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

The norm of E is said to be *Gâteaux differentable* (and E is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{3}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (3) is attained uniformly for $(x, y) \in U \times U$.

The (normalized) *duality* mapping J from E into the family of non-empty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : (x, f) = \|x\|^2 = \|f\|^2\}.$$

for each $x \in E$. It is single valued if and only if E is smooth. It is also well-known that if E has a uniformly Fréchet differentiable norm, J is uniformly continuous on bounded subsets of E. (cf. [3, 5]). Suppose that J is single valued. Then J is said to be *weakly sequentially continuous* if for each $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J(x_n) \overset{*}{\rightharpoonup} J(x)$.

A Banach space E is said to satisfy *Opial's condition* [10] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. We know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition; see [7].

Recall that a mapping T defined on a subset C of a Banach space E (and taking values in E) is said to be *demiclosed* if for any sequence $\{u_n\}$ in C the following implication holds:

$$u_n \rightharpoonup u \text{ and } \lim_{n\to\infty} \|Tu_n - w\| = 0$$

implies

$$u \in C \text{ and } Tu = w.$$

The following lemma can be found in [6, p. 108].

LEMMA 1. *Let E be a reflexive Banach space which satisfies Opial's condition, let C be a nonempty closed convex subset of E, and suppose $T : C \to E$ is nonexpansive. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping.*

Finally, let C be a nonempty closed convex subset of E. A mapping Q of C into C is said to be a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is range of Q. Let D be a subset of C and let Q be a mapping of C into D. Then Q is said to be *sunny* if each point on the ray $\{Qx + t(x - Qx) \ : t > 0\}$ is mapped by Q back onto Qx, in other words,

$$Q(Qx + t(x - Qx)) = Qx$$

for all $t \geq 0$ and $x \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D; for more details, see [5].

The following lemma is well-known (cf. [5, p. 48]).

LEMMA 2. *Let C be a nonempty closed convex subset of a smooth Banach space E, D a subset of C, $J : E \to E^*$ the duality mapping of E, and $Q : C \to D$ be a retraction. Then the following are equivalent:*

(a) $(x - Qx, J(y - Qx)) \leq 0$ *for all* $x \in C$ *and* $y \in D$;

(b) $\|Qz - Qw\|^2 \leq (z - w, J(Qz - Qw))$ *for all* z *and* w *in* C;

(c) Q *is both sunny and nonexpansive.*

3. Main results

In this section, we study the strong convergence of $\{x_n\}$ defined by (2) in a uniformly smooth Banach space with a weakly sequentially continuous duality mapping.

From now on, let $\{\lambda_n\}$ be a sequence in $[0,1)$ which satisfies the following:

(A1) $\lim_{n\to\infty}\lambda_n = 0$,

(A2) $\sum_{n=0}^{\infty}\lambda_n = \infty$; equivalently $\prod_{n=1}^{\infty}(1-\lambda_n) = 0$

(A3) $\sum_{n=0}^{\infty}|\lambda_n - \lambda_{n+N}| < \infty$.

As in the Introduction, we set $T_n := T_{n \bmod N}$, where we let the mod N function take values in $\{1,\cdots,N\}$.

THEOREM 1. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$, C a nonempty closed convex subset of E, and $T_1,\cdots,T_N$ nonexpansive mappings from C into itself with $F := \cap_{i=1}^{N} Fix(T_i)$ nonempty and*

$$F = Fix(T_N\cdots T_1) = Fix(T_1T_N\cdots T_3T_2) = \cdots$$
$$= Fix(T_{N-1}T_{N-2}\cdots T_1T_N).$$

Let $\{\lambda_n\}$ be a sequence in $[0,1)$ which satisfies (A1), (A2) and (A3). Then for any a and x_0 in C, the sequence $\{x_n\}$ defined by (2) converges strongly to Q_Fa, where Q is a sunny nonexpansive retraction of C onto F.

We need the following result which is obtained in [1].

LEMMA 3. *Let E be a Banach space and let $C, T_1,\cdots,T_N, F, \{\lambda_n\}$, and $\{x_n\}$ be as in Theorem 1. Then for $x_0 = a$,*

$$\lim_{n\to\infty}\|x_n - T_{n+N}\cdots T_{n+1}x_n\| = 0.$$

Proof. The proof still follows the lines of the proof in [1]. So we omit the proof.

Proof of Theorem 1. First we prove the case $x_0 = a$. Note that $\{x_n\}$ is bounded since $F \neq \emptyset$. In fact, by induction, we show that

$$\|x_n - z\| \le \|a - z\| \tag{4}$$

for all $n \geq 0$ and $z \in F$. Let $z \in F$. Clearly, (4) holds for $n = 0$. If $\|x_n - z\| \leq \|a - z\|$, then we have

$$\begin{aligned}\|x_{n+1} - z\| &\leq \lambda_{n+1}\|a - z\| + (1 - \lambda_{n+1})\|T_{n+1}x_n - z\| \\ &\leq \lambda_{n+1}\|a - z\| + (1 - \lambda_{n+1})\|x_n - z\| \\ &\leq \|a - z\|.\end{aligned}$$

Let a subsequence $\{x_{n'}\}$ of $\{x_n\}$ be such that

$$\lim_{n' \to \infty}(a - Q_F a, J(x_{n'+1} - Q_F a)) = \limsup_{n \to \infty}(a - Q_F a, J(x_{n+1} - Q_F a)).$$

We assume (after passing to another subsequence if necessary) that $n' + 1 \bmod N = i$ for some $i \in \{1, \cdots, N\}$ and that $x_{n'+1} \rightharpoonup x$. From Lemma 3, it follows that $\lim_{n' \to \infty}\|x_{n'+1} - T_{i+N} \cdots T_{i+1}x_{n'+1}\| = 0$. Hence, by Lemma 1, we have $x \in Fix(T_{i+N} \cdots T_{i+1}) = F$. On the other hand, since E is uniformly smooth, F is a sunny nonexpansive retract of C (cf. [5, p. 49]). Thus, by weakly sequentially continuity of duality mapping J and Lemma 2, we have

$$\begin{aligned}&\limsup_{n \to \infty}(a - Q_F a, J(x_{n+1} - Q_F a)) \\ &= \lim_{n' \to \infty}(a - Q_F a, J(x_{n'+1} - Q_F a)) \\ &= (a - Q_F a, J(x - Q_F a)) \leq 0.\end{aligned} \tag{5}$$

Since $(1 - \lambda_{n+1})(T_{n+1}x_n - Q_F a) = (x_{n+1} - Q_F a) - \lambda_{n+1}(a - Q_F a)$, by using the inequality $\|u\|^2 - \|v\|^2 \geq 2(u - v, J(v))$ for all $u, v \in E$ with $u = (1 - \lambda_{n+1})(T_{n+1}x_n - Q_F a)$ and $v = (x_{n+1} - Q_F a)$, we have

$$\begin{aligned}&\|x_{n+1} - Q_F a\|^2 \\ &\leq (1 - \lambda_{n+1})^2\|(T_{n+1}x_n - Q_F a)\|^2 \\ &\qquad + 2\lambda_{n+1}(a - Q_F a, J(x_{n+1} - Q_F a)) \\ &\leq (1 - \lambda_{n+1})\|x_n - Q_F a\|^2 \\ &\qquad + 2\lambda_{n+1}(a - Q_F a, J(x_{n+1} - Q_F a)).\end{aligned} \tag{6}$$

Now, let $\varepsilon > 0$ be arbitrary. Then by (5), there exists n_ε such that

$$(a - Q_F a, J(x_{n+1} - Q_F a)) \leq \varepsilon \quad \text{for all } n \geq n_\varepsilon.$$

Thus, from (6), we have

$$\|x_{n+1} - Q_F a\|^2 \leq (1 - \lambda_{n+1})\|x_n - Q_F a\|^2 + 2\lambda_{n+1}\varepsilon$$

and hence inductively

$$\|x_{n+1} - Q_F a\|^2 \leq 2\varepsilon + \|x_{n_\varepsilon} - Q_F a\|^2 \prod_{k=n_\varepsilon+1}^{n+1} (1 - \lambda_k) \quad \text{for all } n \geq n_\varepsilon.$$

Letting $n \to \infty$, we have

$$\limsup_{n \to \infty} \|x_n - Q_F a\|^2 \leq 2\varepsilon.$$

Since ε was arbitrary, $\{x_n\}$ converges strongly to $Q_F a$.

Next, let x_0 be arbitrary (possibly different from a) and let $\{y_n\}$ be the sequence with starting point $y_0 := a$. Then, by the above fact, we have

$$\lim_{n \to \infty} y_n = Q_F a.$$

On the other hand, it is easy to check that

$$\|x_n - y_n\| \leq \|x_0 - y_0\| \prod_{k=1}^{n} (1 - \lambda_k) \quad \text{for all } n \geq 0.$$

Thus we have $\lim_{n \to \infty} \|x_n - y_n\| = 0$ and hence $\{x_n\}$ converges strongly to $Q_F a$.

This completes the proof.

As direct consequences, we have the following:

COROLLARY 1 (Bauschke [1, Theorem 3.1]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \cap_{i=1}^{N} Fix(T_i)$ nonempty and*

$$F = Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots$$
$$= Fix(T_{N-1} T_{N-2} \cdots T_1 T_N).$$

Let $\{\lambda_n\}$ be a sequence in $[0,1)$ which satisfies (A1), (A2) and (A3). Then for any a and x_0 in C, the sequence $\{x_n\}$ defined by (2) converges strongly to $P_F a$, where P is the nearest point projection of C onto F.

Proof. Note that the nearest point projection P of C onto F is a sunny nonexpansive retraction. Thus the result follows from Theorem 1.

COROLLARY 2 ([Wittmann [13, Theorem 2]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and T a nonexpansive mapping from C into itself with $Fix(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $[0,1)$ which satisfies* (A1), (A2) *and* (A3). *Then for any a and x_0 in C, the sequence $\{x_n\}$ defined by* (2) *(with $N = 1$) converges strongly to $P_{Fix(T)}a$, where P is the nearest point projection of C onto $Fix(T)$.*

Let D be a subset of a Banach space E. Recall that a mapping $T : D \to E$ is said to to be firmly nonexpansive if for each x and y in D, the convex function $\phi : [0,1] \to [0,\infty)$ defined by

$$\phi(s) = \|(1-s)x + sTx - ((1-s)y + sTy)\|$$

is nonincreasing. Since ϕ is convex, it is easy to check that a mapping $T : D \to E$ is firmly nonexpansive if and only if

$$\|Tx - Ty\| \leq \|(1-t)(x-y) + t(Tx - Ty)\|$$

for each x and y in D and $t \in [0,1]$. It is clear that every firmly nonexpansive mapping is nonexpansive (cf. [5, 6]).

The following result extends a Lions-type iteration scheme [9] to Banach spaces.

COROLLARY 3. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$, C a nonempty closed convex subset of E, and $T_1, \cdots, T_N$ firmly nonexpansive mappings from C into itself with $F := \cap_{i=1}^N Fix(T_i)$ nonempty and*

$$\begin{aligned} F = Fix(T_N \cdots T_1) &= Fix(T_1 T_N \cdots T_3 T_2) = \cdots \\ &= Fix(T_{N-1} T_{N-2} \cdots T_1 T_N). \end{aligned}$$

Let $\{\lambda_n\}$ be a sequence in $[0,1)$ which satisfies (A1), (A2) *and* (A3). *Then for any a and x_0 in C, the sequence $\{x_n\}$ defined by* (2) *converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto F.*

REMARK. (1) In Hilbert space, Lions [9] had used

(L1) $\lim_{n\to\infty} \lambda_n = 0$,

(L2) $\sum_{k=1}^{\infty} \lambda_{kN+i} = \infty$ for all $i = 0, \cdots, N-1$,

which is more restrictive than (A2), and

(L3) $\lim_{k\to\infty} \frac{\sum_{i=1}^{N} |\lambda_{kN+i} - \lambda_{(k-1)N+i}|}{(\sum_i^N \lambda_{kN+i})^2} = 0$

in place of (A3).

(2) In general, (A3) and (L2) are independent, even when $N = 1$: if $\lambda_n := \frac{1}{n+1}$, then $\{\lambda_n\}$ satisfies (A3) and fails (L3). In contrast, if $\{\lambda_n\}$ is given by $\lambda_{2n} := (n+1)^{-\frac{1}{4}}$ and $\lambda_{2n+1} := (n+1)^{-\frac{1}{4}} + (n+1)^{-1}$, then (L3) holds but (A3) does not. For more details, see [1].

References

1. H. H. Bauschke, *The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space*, J. Math. Anal. Appl. **202** (1996), 150–159.
2. F. E. Browder, *Convergence of approximations to fixed points of nonexpansive mappings in Banach spaces*, Archs Ration. Mech. Anal. **24** (1967), 82–90.
3. J. Diestel, *Geometry of Banach Spaces*, Lectures Notes in Math. 485, Springer-Verlag, Berlin, Heidelberg, 1975.
4. A. Genel and J. Lindenstrauss, *An example concerning fixed points*, Israel J. Math. **22** (1975), 81–86.
5. K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.
6. K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, in "Cambridge Studies in Advanced Mathematics," Vol. 28, Cambridge Univ. Press, Cambridge, UK, 1990.
7. J. P. Gossez and E. L. Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math. **40(3)** (1972), 565–573.
8. B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.
9. P. L. Lions, *Approximation de points fixes de contractions*, C. R. Acad. Sci. Sér A-B, Paris **284** (1977), 1357–1359.
10. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.
11. S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.
12. N. Shioji and W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, preprint.

13. R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math **59** (1992), 486–491.

Jong Soo Jung
Department of Mathematics, Dong-A University, Pusan 604-714, Korea
E-mail: jungjs@seunghak.donga.ac.kr

Tae Hwa Kim
Department of Applied Mathematics, Pukyong National University, Puasn 608-737, Korea
E-mail: taehwa@dolphin.pknu.ac.kr