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REGULARITY OF SOLUTIONS TO HELMHOLTZ–TYPE PROBLEMS WITH ABSORBING BOUNDARY CONDITIONS IN NONSMOOTH DOMAINS

Jinsoo Kim and Dongwoo Sheen

1. Introduction

For the numerical simulation of wave phenomena either in unbounded domains or in so large domains that it is not feasible to compute solutions on the entire region, it is needed to truncate the original domains to manageable bounded domains whose geometries are simple but usually nonsmooth. On the artificial boundaries thus created, absorbing boundary conditions are taken so that the significant part of waves arriving at the artificial boundaries can be transmitted [5, 10, 11, 16, 17, 26].

In order to solve initial boundary value problems numerically, usual numerical methods require the problems for the previous time steps to be solved before going into the next time step. Instead of solving initial boundary value problems in the time domain, we solve in the frequency domain elliptic problems which are Fourier transforms of original time–dependent problems with respect to time and then obtain the solution in the time domain by the Fourier's inversion formula [8, 9]. We note that the Fourier transformation of wave problem with respect to time generates a family of Helmholtz–type problems which, when subject to absorbing boundary conditions, are uniquely solvable elliptic problems for each nonzero frequency and can be solved simultaneously for all frequencies.

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In this approach, regularity of the solutions in the frequency domain plays an important role in error analysis as most of the error estimates for the numerical solution to elliptic boundary value problem rely on the shift theorem which is the main feature of elliptic boundary value problems in smooth domains [4, 7, 21]. It is important to extend the results for smooth domains to the corresponding ones for nonsmooth domains which are usually Lipschitz domains [6, 15, 25]. Indeed, under the assumption of $H^2(\Omega)$–regularity of the solutions in the frequency domain, we obtained the error estimates of optimal order in the time domain [12].

In this paper, we prove the shift theorem for Helmholtz–type equations with Robin boundary conditions in bounded convex domains whose coefficients are complex–valued functions. In the proof, we approximate the nonsmooth domain by a sequence of domains with C^2–boundaries and use a priori estimates which are independent of the domains and of the boundary conditions [19]. The existence and the uniqueness of the solution are also given.

2. Preliminaries and Notations

Let Ω be a bounded domain in $\mathbb{R}^n$ which is an artificial truncation of the original medium and set $J = (0, \infty)$. Consider the time–dependent wave equation:

$$\frac{1}{c(x)^2}u_{tt} - \Delta u = f(x,t), \qquad (x,t) \in \Omega \times J, \tag{2.1a}$$

$$\frac{1}{c(x)}u_t + u_\nu = 0, \qquad (x,t) \in \Gamma \times J, \tag{2.1b}$$

$$u|_{t=0} = u_t|_{t=0} = 0, \qquad x \in \Omega, \tag{2.1c}$$

where the coefficient $c \in C^1(\overline{\Omega})$ denotes the wave speed and ν the outward unit normal vector on the boundary $\Gamma = \partial\Omega$. The boundary condition (2.1b) is a standard first–order absorbing boundary condition which makes the artificial boundary Γ transparent to normally outgoing waves so that the significant parts of waves arriving normally at the boundary is completely absorbed [5, 10, 11, 16].

We reformulate the space–time formulation of the wave equation in the space–frequency domain by taking the Fourier transformation of (2.1a)–(2.1b). The problem becomes to solve a family of elliptic problems for $\widehat{u}(\cdot,\omega)$, the Fourier transform of $u(\cdot,t)$ with respect to t, which is defined as usual by

$$\widehat{u}(x,\omega)=\int_{-\infty}^{\infty}u(x,t)e^{-i\omega t}\,dt.$$

Since $u(x,t)$ is a real-valued function, its Fourier transform satisfies the conjugation relation

$$\widehat{u}(x,-\omega)=\overline{\widehat{u}}(x,\omega), \tag{2.2}$$

for all $\omega\in\mathbb{R}$. The Fourier's inversion formula recovers u by

$$\begin{aligned}u(x,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{u}(x,\omega)e^{i\omega t}\,d\omega\\&=\frac{1}{\pi}\operatorname{Re}\int_{0}^{\infty}\widehat{u}(x,\omega)e^{i\omega t}\,d\omega.\end{aligned}$$

We shall consider f and u in (2.1a)–(2.1c) to be extended to $t<0$ by zero. Taking the Fourier transformation of the equations (2.1a)–(2.1b) with respect to t leads to the elliptic problems:

$$-\frac{\omega^2}{c(x)^2}\widehat{u}-\Delta\widehat{u}=\widehat{f}(x,\omega),\qquad (x,\omega)\in\Omega\times\mathbb{R}, \tag{2.3a}$$

$$\frac{i\omega}{c(x)}\widehat{u}+\widehat{u}_\nu=0,\qquad (x,\omega)\in\Gamma\times\mathbb{R}. \tag{2.3b}$$

For the waves with attenuation, equation (2.3a)–(2.3b) should be modified in order to take into account friction. We generalize the equation (2.3a) by including a dissipative term and deduce the corresponding absorbing boundary condition using dispersion relation [9].

$$-\frac{\omega^2}{c(x)^2}\widehat{u}+i\omega b(x,\omega)\widehat{u}-\Delta\widehat{u}=\widehat{f}(x,\omega),\quad (x,\omega)\in\Omega\times\mathbb{R}, \tag{2.4a}$$

$$i\alpha(x,\omega)\widehat{u}+\widehat{u}_\nu=0,\qquad (x,\omega)\in\Gamma\times\mathbb{R}, \tag{2.4b}$$

where

$$\alpha(x,\omega) = \frac{\omega}{\sqrt{2}c(x)}\left\{1+\left(1+\frac{b(x,\omega)^2}{\omega^2}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}} - i\frac{b(x,\omega)}{\sqrt{2}c(x)}\left\{1+\left(1+\frac{b(x,\omega)^2}{\omega^2}\right)^{\frac{1}{2}}\right\}^{-\frac{1}{2}}$$

and b is a real nonnegative generalized friction coefficient. We assume that $b(\cdot,\omega) = b(\cdot,-\omega) \geq 0$ and $b(\cdot,\omega) \in C^1(\overline{\Omega})$ for all $\omega \in \mathbb{R}$ and that

$$\omega b(\cdot,\omega) \to 0 \qquad \text{as } \omega \to 0,$$
$$\omega^{-1} b(\cdot,\omega) \to 0 \qquad \text{as } \omega \to 0.$$

Note that since the boundary condition (2.4b) is nonlocal in both space and time, the formulation in the time domain leads to a pseudodifferential problem and is not useful for practical calculations.

Since the source function $f(x,t)$ is a real-valued function, an application of (2.2) to the equations, (2.3a)–(2.3b) and (2.4a)–(2.4b) leads to $\widehat{u}(\cdot,-\omega) = \overline{\widehat{u}}(\cdot,\omega)$. Therefore, it suffices to find the solution $\widehat{u}(\cdot,\omega)$ for all $\omega \geq 0$, after which the solution $u(\cdot,t)$ in the time domain is found by the Fourier's inversion formula.

In the case of $\omega = 0$, the equation (2.3a)–(2.3b) becomes a Neumann problem and admits a solution which is unique up to an additive constant if

$$\int_\Omega \widehat{f}(x,0)\,dx = \int_\Omega \int_0^\infty f(x,t)\,dt\,dx = 0.$$

The same is also true for the equation (2.4a)–(2.4b).

Let us assume that the truncated inhomogeneous medium is surrounded by a homogeneous medium, that is, there exists an open subset Ω_0 of $\mathbb{R}^n$ such that $\Omega \Subset \Omega_0$ and the coefficients $c(\cdot)$ and $b(\cdot,\omega)$ are constants in $\Omega_0 \setminus \Omega$ for all $\omega > 0$.

We concentrate on the case of $\omega > 0$ and in the next section we deal with a general Helmholtz–type problem with Robin boundary condition.

All functions are assumed to have their values in $\mathbb{C}$ and standard notations for function spaces and their norms will be used in this paper [1, 7, 14, 15]. For a given domain D, we denote by $\|\cdot\|_{m,p,D}$ the norm of the Sobolev space $W^{m,p}(D)$. If $p = 2$, $W^{m,p}(D)$ will be denoted by

$H^m(D)$. By $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$ we denote L^2–inner products in $L^2(D)$ and $L^2(\partial D)$, respectively. When $m = 0$ or $p = 2$, the subscripts m or p may be dropped. Furthermore, whenever the domain D is obviously understood in the context, the subscript D will be omitted.

3. Regularity

Let Ω be a convex bounded domain in $\mathbb{R}^n$. Consider the problem

$$-\Delta u + \lambda u = f \qquad \text{in } \Omega, \tag{3.1a}$$

$$\frac{\partial u}{\partial \nu} + \mu u = 0 \qquad \text{on } \Gamma, \tag{3.1b}$$

where $\lambda = \lambda(x)$ is a bounded measurable complex-valued function and μ is a nonzero complex number with a nonnegative real part and a positive imaginary part.

We assume that there exists an open subset Ω_0 of $\mathbb{R}^n$ such that $\Omega \Subset \Omega_0$, in which λ is a C^1–continuous function with a nonnegative imaginary part and that there exists a point p on the analytic portion of Γ and a neighborhood of p in which λ is analytic.

Define a sesquilinear form $\Lambda(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\Lambda(u, v) = (\nabla u, \nabla v) + (\lambda u, v) + \langle \mu u, v\rangle_\Gamma. \tag{3.2}$$

The weak formulation of the equation (3.1a)–(3.1b) is given as follows: *Find $u \in H^1(\Omega)$ such that*

$$\Lambda(u, v) = (f, v) \qquad \textit{for all } v \in H^1(\Omega). \tag{3.3}$$

We then have the following classical results for the regular elliptic problem. For the precise proof of the uniqueness, refer to [8] and [9].

THEOREM 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $f \in L^2(\Omega)$. Then there exists a unique solution $u \in H^1(\Omega)$ to the equation (3.1a)–(3.1b). Moreover, if Ω is of class C^2, $u \in H^2(\Omega)$ and we have*

$$\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}\,,$$

where C is a positive constant which is independent of f.

Proof. Cauchy-Kowalevsky theorem, unique continuation principle and local regularity imply the uniqueness of the solution [20]. The existence then follows from Fredholm alternative [7]. By an a priori estimate of Agmon–Douglis–Nirenberg [2, 3] $H^2(\Omega)$–regularity is obtained. □

The following a priori estimates will play a fundamental role in the proof of the regularity in nonsmooth domains [19].

THEOREM 3.2. *Let Ω be a convex bounded domain in $\mathbb{R}^n$ with a C^2 boundary and λ a bounded measurable function such that $\operatorname{Re}\lambda \geq \lambda_0 > 0$. Then we have*

$$\|u\|_{2,\Omega} \leq C(\lambda) \, \| -\Delta u + \lambda u \, \|_{0,\Omega} \, ,$$

for all $u \in H^2(\Omega)$ satisfying the boundary condition (3.1b), *where the constant $C(\lambda)$ is given by*

$$C(\lambda) = \left\{ \frac{1}{\lambda_0}\left(1 + \frac{1}{\lambda_0}\right) + \left(1 + \frac{\|\lambda\|_{\infty,\Omega}}{\lambda_0}\right)^2 \right\}^{\frac{1}{2}} .$$

The following lemma enables us to approximate nonsmooth domain by a sequence of smooth domains [15, 23, 24, 27].

LEMMA 3.1. *Let Ω be a convex bounded domain in $\mathbb{R}^n$. Then there exists a sequence, $\{\Omega_m\}_{m=1}^{\infty}$, of convex bounded domain in $\mathbb{R}^n$ with C^2 boundaries Γ_m such that $\Omega \subset \Omega_m$ and $d(\Gamma, \Gamma_m) \to 0$ as $m \to \infty$. For large enough m, there exists a finite number of open subsets V_k, $k = 1, 2, \ldots, K$, of $\mathbb{R}^n$ with the following properties:*

1. *For each $k \in \mathbb{N}$, there exist local coordinates $\{z_1^k, \ldots, z_n^k\}$ in which V_k is the hypercube*

$$\{(z_1^k, \ldots, z_n^k) \mid -a_j^k < z_j^k < a_j^k, \quad 1 \leq j \leq n\}.$$

2. *For each $k \in \mathbb{N}$, there exist Lipschitz functions φ^k and φ_m^k defined in*

$$V_k' = \{z_k' = (z_1^k, \ldots, z_{n-1}^k) \mid -a_j^k < z_j^k < a_j^k, \quad 1 \leq j \leq n-1\}$$

such that

$$|\varphi^k(z_k')|, \, |\varphi_m^k(z_k')| < \frac{a_n^k}{2} \qquad \text{for every } z_k' \in V_k',$$

$$
\begin{aligned}
\Omega \cap V_k &= \{z^k = (z_k', z_n^k) \mid z_n^k > \varphi^k(z_k')\}, \\
\Gamma \cap V_k &= \{z^k = (z_k', z_n^k) \mid z_n^k = \varphi^k(z_k')\}, \\
\Omega_m \cap V_k &= \{z^k = (z_k', z_n^k) \mid z_n^k > \varphi_m^k(z_k')\}, \\
\Gamma_m \cap V_k &= \{z^k = (z_k', z_n^k) \mid z_n^k = \varphi_m^k(z_k')\}.
\end{aligned}
$$

3. $\Gamma \subset \bigcup_{k=1}^{K} V_k$ *and* $\Gamma_m \subset \bigcup_{k=1}^{K} V_k$.
   In addition φ^k *is a convex function and* φ_m^k *is a* C^2*–continuous convex function for all large enough* $m \in \mathbb{N}$.
4. $\varphi_m^k \to \varphi^k$ *uniformly on* V_k' *and there exists a positive constant* L *such that*

$$
|\nabla \varphi^k(z_k')|,\ |\nabla \varphi_m^k(z_k')| \le L,
$$

for every $z_k' \in V_k'$, $1 \le k \le K$. *Furthermore,* $\nabla \varphi_m^k \to \nabla \varphi^k$ *a.e. in* V_k'.

Now we state the main theorem.

THEOREM 3.3. *Let* Ω *be a convex bounded domain in* $\mathbb{R}^n$ *and let* $f \in L^2(\Omega)$. *Then there exists a unique solution* $u \in H^2(\Omega)$ *to the equation* (3.1a)–(3.1b). *Moreover, we have*

$$
\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega},
$$

where C *is a positive constant which is independent of* f.

To prove Theorem 3.3, we approximate the original domain Ω from the outside by a sequence of domains Ω_m with smoother boundaries Γ_m which was chosen as in Lemma 3.1, and then consider a sequence of the solutions u_m to the equation (3.1a)–(3.1b) with respect to Ω_m and Γ_m. In Theorem 3.1, it is required that there exists a point p_m in the analytic portion of Γ_m and a neighborhood of p_m in which λ is real analytic for the well–posedness. These conditions will be satisfied provided that in the neighborhood of the point $p \in \Gamma$, the analytic portion of Γ can be approximated by the analytic portion of Γ_m for sufficiently large m. In fact, by the assumptions on λ and Ω_0, this construction of Ω_m is possible.

Proof. Let Ω_m be chosen as in Lemma 3.1 and let f_m be the extension of f to $\Omega_m \setminus \Omega$ by zero. For each $m \in \mathbb{N}$, consider the following problem:

$$\begin{cases} -\Delta u_m + \lambda u_m = f_m & \text{in } \Omega_m, \\ \dfrac{\partial u_m}{\partial \nu} + \mu u_m = 0 & \text{on } \Gamma_m \end{cases} \tag{3.4}$$

Then by Theorem 3.1, there exists a unique solution u_m to the equation (3.4) for each $m \in \mathbb{N}$.

Due to the compact perturbations [13, 18, 22], we may assume that $\operatorname{Re}\lambda \geq \lambda_0 > 0$. By Theorem 3.2 there exists a subsequence of $\{u_m\}$, which will be denoted again by $\{u_m\}$, such that

$$u_m \rightharpoonup u \quad \text{weakly in } H^2(\Omega) \quad \text{as } m \to \infty.$$

We are to show that u is the solution to the equation (3.1a)–(3.1b). For each $v \in H^1(\Omega)$, there exists a Calderón extension $\tilde{v} \in H^1(\mathbb{R}^n)$ of v such that $\tilde{v}|_\Omega = v$[1]. We therefore have, for all $m \in \mathbb{N}$,

$$(\nabla u_m, \nabla \tilde{v})_{\Omega_m} + (\lambda u_m, \tilde{v})_{\Omega_m} + \langle \mu u_m, \tilde{v}\rangle_{\Gamma_m} = (f, \tilde{v})_{\Omega_m}. \tag{3.5}$$

Now, each term in (3.5) will be shown to converge to the corresponding term in (3.3) as $m \to \infty$. First, we have

$$\begin{aligned} |(\lambda u_m, \tilde{v})_{\Omega_m} - (\lambda u, v)_\Omega| &\leq \left|(\lambda u_m, \tilde{v})_{\Omega_m \setminus \Omega}\right| + |(\lambda(u_m - u), v)_\Omega| \\ &\leq \|\lambda u_m\|_{0,\Omega_m} \|\tilde{v}\|_{0,\Omega_m \setminus \Omega} + \|\lambda(u_m - u)\|_{0,\Omega} \|v\|_{0,\Omega} \\ &\leq C\left(\|\tilde{v}\|_{0,\Omega_m \setminus \Omega} + \|u_m - u\|_{0,\Omega} \|v\|_{0,\Omega}\right). \end{aligned}$$

Both terms in the right-hand side in the above inequality converges to zero as $m \to \infty$ by Lebesgue dominated convergence theorem and Rellich–Kondrachov theorem. Hence we obtain

$$(\lambda u_m, \tilde{v})_{\Omega_m} \to (\lambda u, v)_\Omega \quad \text{as } m \to \infty.$$

In the same way, we get $(\nabla u_m, \nabla \tilde{v})_{\Omega_m} \to (\nabla u, \nabla v)_\Omega$ as $m \to \infty$ by Rellich–Kondrachov theorem.

It remains to show

$$\langle u_m, \tilde{v}\rangle_{\Gamma_m} \to \langle u, v\rangle_\Gamma \qquad \text{as } m \to \infty.$$

Let us choose a C^∞–partition of unity, $\{\omega_k\}$ subordinate to $\{V_k\}$. It is sufficient to show

$$\langle \omega_k u_m, \tilde{v}\rangle_{\Gamma_m} \to \langle \omega_k u, v\rangle_\Gamma \quad \text{as } m \to \infty,$$

for all $k = 1, 2, \ldots, K$. By change of variables, we have

$$\int_{\Gamma_m} \omega_k u_m \overline{\tilde{v}}\, d\sigma_m = \int_{V_k'} (\omega_k u_m \overline{\tilde{v}})(z_k', \varphi_m^k(z_k'))\{1 + |\nabla \varphi_m^k(z_k')|^2\}^{\frac{1}{2}}\, dz_k'$$

and

$$\int_{\Gamma} \omega_k u\, \overline{v}\, d\sigma = \int_{V_k'} (\omega_k u\, \overline{v})(z_k', \varphi^k(z_k'))\{1 + |\nabla \varphi^k(z_k')|^2\}^{\frac{1}{2}}\, dz_k'.$$

Thus, we obtain

$$\langle \omega_k u_m, \tilde{v}\rangle_{\Gamma_m} - \langle \omega_k u, v\rangle_{\Gamma} \equiv I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{V_k'} (\omega_k u\, \overline{v})(z_k', \varphi^k(z_k')) \left\{ \left(1 + |\nabla \varphi_m^k(z_k')|^2\right)^{\frac{1}{2}} - \left(1 + |\nabla \varphi^k(z_k')|^2\right)^{\frac{1}{2}} \right\} dz_k'$$

$$I_2 = \int_{V_k'} \left\{ (\omega_k u_m \overline{\tilde{v}})(z_k', \varphi_m^k(z_k')) - (\omega_k u_m \overline{\tilde{v}})(z_k', \varphi^k(z_k')) \right\} \left(1 + |\nabla \varphi_m^k(z_k')|^2\right)^{\frac{1}{2}} dz_k'$$

$$I_3 = \int_{V_k'} \left\{ (\omega_k u_m \overline{\tilde{v}})(z_k', \varphi^k(z_k')) - (\omega_k u\, \overline{v})(z_k', \varphi^k(z_k')) \right\} \left(1 - |\nabla \varphi_m^k(z_k')|^2\right)^{\frac{1}{2}} dz_k'.$$

Now,

$$|I_1| \leq 2(1 + L^2)^{\frac{1}{2}} \int_{V_k'} \left|(\omega_k u\, \overline{v})(z_k', \varphi^k(z_k'))\right| dz_k' < \infty.$$

Thus by Lebesgue dominated convergence theorem, $I_1 \to 0$ as $m \to \infty$.

For I_2, without loss of generality, we shall assume that $\tilde{v} \in C_0^1(\mathbb{R}^n)$. Using the relation

$$\begin{aligned}(\omega_k u_m \overline{\tilde{v}})(z_k', \varphi_m^k(z_k')) &- (\omega_k u_m \overline{\tilde{v}})(z_k', \varphi^k(z_k')) \\ &= \int_{\varphi^k(z_k')}^{\varphi_m^k(z_k')} \frac{\partial}{\partial z_n} (\omega_k u_m \overline{\tilde{v}})(z_k', z_n^k)\, dz_n^k,\end{aligned}$$

we obtain

$$
\begin{aligned}
|I_2| &\le \left\{ \int_{V_k'} \left| \left(1 + |\nabla \varphi_m^k(z_k')|^2\right) \right| dz_k' \right\}^{\frac{1}{2}} \\
&\quad \cdot \left\{ \int_{V_k'} \left| (\omega_k u_m \overline{\tilde{v}})(z_k', \varphi_m^k(z_k')) - (\omega_k u_m \overline{\tilde{v}})(z_k', \varphi^k(z_k')) \right|^2 dz_k' \right\}^{\frac{1}{2}} \\
&\le C(1 + L^2)^{\frac{1}{2}} \left\{ \int_{\Omega_m \cap V_k} \left| \frac{\partial}{\partial z_n} (\omega_k u_m \overline{\tilde{v}}) \right|^2 dz^k \right\}^{\frac{1}{2}} \|\varphi_m^k - \varphi^k\|_{\infty, V_k'}^{\frac{1}{2}} \\
&\le C \|\omega_k \tilde{v}\|_{1,\infty,\Omega_m \cap V_k} \|u_m\|_{1,\Omega_m \cap V_k} \|\varphi_m^k - \varphi^k\|_{\infty, V_k'}^{\frac{1}{2}}.
\end{aligned}
$$

Hence we get

$$
|I_2| \le C \|\varphi_m^k - \varphi^k\|_{\infty, V_k'}^{\frac{1}{2}} \to 0 \quad \text{as } m - \infty.
$$

For I_3, we have

$$
\begin{aligned}
|I_3| &\le (1 + L^2)^{\frac{1}{2}} \int_{V_k'} \left| (\omega_k u_m \overline{\tilde{v}} - \omega_k u\, \overline{v})(z_k', \varphi^k(z_k')) \right| dz_k' \\
&\le (1 + L^2)^{\frac{1}{2}} \|u_m - u\|_{0,\Gamma} \|\omega_k v\|_{0,\Gamma} \to 0 \qquad \text{as } m \to \infty.
\end{aligned}
$$

Therefore u satisfies the equation (3.3) and we have

$$
\|u\|_{2,\Omega} \le C(\lambda) \|f\|_{0,\Omega}.
$$

This completes the proof. □

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DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151–742, KOREA
E-mail: jskim@math.snu.ac.kr
E-mail: sheen@math.snu.ac.kr