Bull. Korean Math. Soc. **34** (1997), No. 4, pp. 549–560

THE DEFORMATION SPACE OF REAL PROJECTIVE STRUCTURES ON THE ($^*n_1n_2n_3n_4$)-ORBIFOLD

JUNGKEUN LEE

ABSTRACT. For positive integers $n_i \geq 2, i = 1, 2, 3, 4$, such that $\sum \frac{1}{n_i} < 2$, there exists a quadrilateral $\mathcal{P} = P_1P_2P_3P_4$ in the hyperbolic plane $\mathbb{H}^2$ with the interior angle $\frac{\pi}{n_i}$ at P_i. Let $\Gamma \subset Isom(\mathbb{H}^2)$ be the (discrete) group generated by reflections in each side of $\mathcal{P}$. Then the quotient space $\mathbb{H}^2/\Gamma$ is a differentiable orbifold of type ($^*n_1n_2n_3n_4$). It will be shown that the deformation space of $\mathbb{R}\mathrm{P}^2$-structures on this orbifold can be mapped continuously and bijectively onto the cell of dimension $4 - |\{i|n_i = 2\}|$.

1. Introduction

Goldman showed that the deformation space of reflection groups in the convex k-gons of type $(n_1n_2\cdots n_k)$ in the projective plane is homeomorphic to the cell of dimension $3k - 8 - |\{i|n_i = 2\}|$ for $k \geq 4$, $n_i \geq 2$, $\sum \frac{1}{n_i} < k - 2$. (See Goldman [2], pp 58-64.) We note that it is similar to"the deformation space of real projective structures on the ($^*n_1n_2\cdots n_k$)-orbifold". (The definition of ($^*n_1n_2\cdots n_k$)-orbifold will be given in the next section.) We will consider the special case $k = 4$: We will define the deformation space of real projective structures on the ($^*n_1n_2n_3n_4$)-orbifold and show that it can be mapped continuously and bijectively onto the cell of dimension $4 - |\{i|n_i = 2\}|$ by concrete matrix calculations, using the deformation theorem in Goldman [3]. But the restriction $k = 4$ is actually unnecessary. Our method can be used equally

Received August 16, 1996.

1991 Mathematics Subject Classification: 57N50.

Key words and phrases: deformation space, real projective structure, ($^*n_1n_2n_3n_4$)-orbifold.

This work was partially supported by GARC-KOSEF(1996).

well in the cases $k > 4$. We restrict ourselves to the case $k = 4$ only to avoid some complications.

2. Orbifolds

This section will be devoted to some basic definitions and propositions about orbifolds. See Ratcliffe [5], Scott [6], or Thurston [7] for more details. Roughly speaking, an orbifold is a topological space locally modeled on open subsets of $\mathbb{R}^n$ quotient out by some finite groups. More precisely,

DEFINITION 1. An *n-orbifold* $\mathcal{O}$ with underlying space $X_{\mathcal{O}}$ is a Hausdorff topological space $X_{\mathcal{O}}$ equipped with a covering by open sets $\{U_i\}$ closed under finite intersection such that

- to each U_i is associated a finite group Γ_i and an action of Γ_i on an open subset $\widehat{U}_i$ of $\mathbb{R}^n$ and a homeomorphism $\phi_i : \widehat{U}_i/\Gamma_i \to U_i$
- whenever $U_i \subset U_j$, there is an inclusion $f_{ij} : \Gamma_i \to \Gamma_j$ and an embedding $\hat{\phi}_{ij} : \widehat{U}_i \to \widehat{U}_j$ equivariant with respect to f_{ij} such that the following diagram commutes :

$$\begin{array}{ccc} \widehat{U}_i & \overset{\hat{\phi}_{ij}}{\rightarrow} & \widehat{U}_j \\ \downarrow & & \downarrow \\ \widehat{U}_i/\Gamma_i & \overset{\phi_{ij}}{\rightarrow} & \widehat{U}_j/f_{ij}\Gamma_i \\ & & \downarrow \\ \downarrow \phi_i & & \widehat{U}_j/\Gamma_j \\ & & \downarrow \\ U_i & \subset & U_j \end{array}$$

DEFINITION 2. The *singular set* $\Sigma_{\mathcal{O}}$ of an orbifold $\mathcal{O}$ is the set of all points x in $X_{\mathcal{O}}$ such that in each local coordinate system $U = \widehat{U}/\Gamma$ near x, and for each $\hat{x}$ in $\widehat{U}$ projecting to x, the stabilizer $\Gamma_{\hat{x}}$ of $\hat{x}$ is nontrivial.

EXAMPLE 3. A manifold without boundary may be regarded as an orbifold whose singular set is empty.

DEFINITION 4. Let $\mathcal{O}$ be a 2-orbifold. A point $x \in X_{\mathcal{O}}$ is a *reflector* if there is a local coordinate $U \to \mathbb{R}^2/\mathbb{Z}_2$ near x where $\mathbb{Z}_2$ acts as the reflection in a line through $0 \in \mathbb{R}^2$ and x corresponds to 0. A point $y \in X_{\mathcal{O}}$ is a *corner reflector of order m* if there is a local coordinate $V \to \mathbb{R}^2/D_{2m}$ near y where D_{2m} acts as the dihedral group of order $2m$ generated by reflections in two lines through 0 which form an angle of size π/m and y corresponds to 0.

EXAMPLE 5. Let $n_1, n_2, \cdots, n_k \geq 2$ be positive integers. The $({}^*n_1n_2 \cdots n_k)$-orbifold is a 2-orbifold with the two-dimensional disk as its underlying space and with the boundary of the disk as the singular set such that

- there are k corner reflectors x_i of order n_i on the boundary lying in the (cyclic) order $x_1, x_2, \cdots, x_k$.
- the other boundary points are reflectors.

DEFINITION 6. A *covering orbifold* of an orbifold $\mathcal{O}$ is an orbifold $\widetilde{\mathcal{O}}$ with a projection $p : X_{\widetilde{\mathcal{O}}} \to X_{\mathcal{O}}$ such that

- p is a local covering; that is, each $\tilde{x} \in X_{\widetilde{\mathcal{O}}}$ has an open neighborhood $\widetilde{U}$ homeomorphic to $\widehat{U}/\Gamma$ (in the sense of above definition) such that $p(\widetilde{U})$ is an open set $\widetilde{U}'$ homeomorphic to $\widehat{U}/\Gamma'$ for some group $\Gamma' \supset \Gamma$ and the following diagram commutes :

$$\begin{array}{ccc} \widehat{U}/\Gamma & \longrightarrow & \widehat{U}/\Gamma' \\ \downarrow & & \downarrow \\ \widetilde{U} & \xrightarrow{p} & \widetilde{U}' \end{array}$$

- p is an even covering, that is, each $x \in X_{\mathcal{O}}$ has an open neighborhood V homeomorphic to $\widehat{V}/\Gamma$ for which each component $\widetilde{U}_j$ of $p^{-1}(V)$ is isomorphic to $\widehat{V}/\Gamma_j$ for some subgroup $\Gamma_j \subset \Gamma$ such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{V}/\Gamma_j & \longrightarrow & \widehat{V}/\Gamma \\ \downarrow & & \downarrow \\ U_j & \xrightarrow{p} & V \end{array}$$

From now on, "covering" will mean orbifold covering.

PROPOSITION 7. *An orbifold has a universal cover. In other words, if $x \in X_{\mathcal{O}} - \Sigma_{\mathcal{O}}$ is a base point for an orbifold $\mathcal{O}$, then there is a (orbifold) covering $p : \widetilde{\mathcal{O}} \to \mathcal{O}$ with base point $\tilde{x}$ (with $p(\tilde{x}) = x$) such that for each* *other covering* $p' : \widetilde{\mathcal{P}} \to \mathcal{O}$ *with base point* $\tilde{x}'$ (*and* $p'(\tilde{x}') = x$), *there is a unique lifting* $q : \widetilde{\mathcal{O}} \to \widetilde{\mathcal{P}}$ *of* p *to a covering of* $\widetilde{\mathcal{P}}$ *with* $q(\tilde{x}) = \tilde{x}'$.

Note that the universal cover is unique; that is, if $x \in X_{\mathcal{O}} - \Sigma_{\mathcal{O}}$ and $p_i : \widetilde{\mathcal{O}}_i \to \mathcal{O}$, $i = 1, 2$, are universal coverings with $p_i(\tilde{x}_i) = x$ then there is a homeomorphism $\alpha : \widetilde{\mathcal{O}}_1 \to \widetilde{\mathcal{O}}_2$ such that α, α^{-1} are coverings with $\alpha(\tilde{x}_1) = \tilde{x}_2$.

DEFINITION 8. Let $p : \widetilde{\mathcal{O}} \to \mathcal{O}$ be an orbifold covering. A *deck transformation* of the covering is a homeomorphism $\gamma : \widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}$ such that $p \circ \gamma = p$.

DEFINITION 9. *The fundamental group* $\pi_1(\mathcal{O})$ of an orbifold $\mathcal{O}$ is the group of deck transformations of the universal covering.

3. The deformation spaces

In this section, we will see the definition of the deformation space of real projective structures on an orbifold.

PROPOSITION 10. *The quotient space of a connected manifold* M *by a group* Γ *which acts faithfully and properly discontinuously on* M *is an orbifold (which we will denote by* M/Γ*). The quotient map* $M \to M/\Gamma$ *is an orbifold covering. If, in addition,* M *is simply connected, then it is the universal covering and* $\pi_1(M/\Gamma)$ *may be identified with* Γ .

Henceforth we will consider only orbifolds of the form $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is a simply connected differentiable manifold without boundary and Γ is a group of diffeomorphisms of $\widetilde{M}$ acting faithfully and properly discontinuously on it.

DEFINITION 11. Let X be a real analytic n-manifold and G a group of analytic diffeomorphisms of it. Let $\mathcal{O}$ be an orbifold $\widetilde{M}/\Gamma$. (So $\Gamma = \pi_1(\mathcal{O})$.) Then a *development pair* of an (X, G)-structure on $\mathcal{O}$ is a pair (dev, H) satisfying the following:

- $dev : \widetilde{M} \to X$ is an immersion.
- $H \in \mathrm{Hom}(\Gamma, G)$: the set of all group homomorphisms of Γ into G.
- H is equivariant with respect to dev, that is, $dev \circ \gamma = H(\gamma) \circ dev : \widetilde{M} \to X$ for each $\gamma \in \Gamma$.

For a development pair (dev, H), dev is called a developing map of the structure and H the holonomy (homomorphism) corresponding to dev.

By the first and third requirements in the above definition, the holonomy is determined by the developing map. That is, if both (dev, H_1) and (dev, H_2) are development pairs then $H_1 = H_2$.

EXAMPLE 12. Let Γ be a subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ acting properly discontinuously on $\mathbb{H}^2$. Since $\mathbb{H}^2$ is simply connected, the quotient map $\mathbb{H}^2 \longrightarrow \mathbb{H}^2/\Gamma$ is the universal orbifold covering. A development pair of a real projective structure (i.e. $(\mathbb{R}\mathrm{P}^2, \mathrm{PGL}(3,\mathbb{R}))$-structure or $\mathbb{R}\mathrm{P}^2$-structure) on this orbifold is a pair (dev, H) such that

- $dev : \mathbb{H}^2 \longrightarrow \mathbb{R}\mathrm{P}^2$ is an immersion.
- $H \in \mathrm{Hom}(\Gamma, \mathrm{PGL}(3,\mathbb{R}))$.
- $dev \circ \gamma = H(\gamma) \circ dev : \mathbb{H}^2 \longrightarrow \mathbb{R}\mathrm{P}^2$ for each $\gamma \in \Gamma$.

Now we can define the deformation space of $\mathbb{R}\mathrm{P}^2$-structures on the orbifold $\mathcal{O} = \widetilde{M}/\Gamma$. For convenience, $\mathrm{PGL}(3,\mathbb{R})$ will be identified with $\mathrm{SL}(3,\mathbb{R})$ and G will be used sometimes in place of them. Let $\mathfrak{D}'(\mathcal{O})$ be the set of all developing maps of $\mathbb{R}\mathrm{P}^2$-structures on $\mathcal{O}$. We topologize $\mathfrak{D}'(\mathcal{O})$ regarding it as a subspace of $C^\infty(\widetilde{M}, \mathbb{R}\mathrm{P}^2)$ with the weak topology. (For the definition of the weak topology, see Hirsch [4].) We also topologize $\mathrm{Hom}(\Gamma, G)$ by the compact-open topology. Since Γ is countable and discrete, the compact-open topology equals the pointwise convergence topology. Assigning to each developing map the holonomy corresponding to it gives a map $hol_1 : \mathfrak{D}'(\mathcal{O}) \longrightarrow \mathrm{Hom}(\Gamma, G)$, which can be shown to be continuous. Fix a point $x_0 \in \mathcal{O}$ and $\tilde{x}_0 \in \widetilde{M}$ projecting to it. Let $Diff_0(\mathcal{O})$ be the identity component in the group $\{\tilde{f} \in Diff(\widetilde{M}) \mid \tilde{f}(\tilde{x}_0) = \tilde{x}_0,\ \tilde{f} \circ \gamma = \gamma \circ \tilde{f} \quad \forall \gamma \in \Gamma\}$. $Diff_0(\mathcal{O})$ acts on $\mathfrak{D}'(\mathcal{O})$ by composition to the right. Let $\mathfrak{D}(\mathcal{O})$ be the quotient space $\mathfrak{D}'(\mathcal{O})/Diff_0(\mathcal{O})$. Since hol_1 is constant on each orbit of the action, it induces $hol_2 : \mathfrak{D}(\mathcal{O}) \longrightarrow \mathrm{Hom}(\Gamma, G)$. Moreover there are actions of G on both $\mathfrak{D}(\mathcal{O})$ and $\mathrm{Hom}(\Gamma, G)$: G acts on $\mathfrak{D}'(\mathcal{O})$ by compositions to the left. Such an action projects to an action of G on $\mathfrak{D}(\mathcal{O})$. On the other hand, G acts on $\mathrm{Hom}(\Gamma, G)$ by conjugations. It can be readily checked that hol_2 induces a well-defined map $hol : \mathfrak{D}(\mathcal{O})/G \longrightarrow \mathrm{Hom}(\Gamma, G)/G$. We denote $\mathfrak{D}(\mathcal{O})/G$ by $\mathfrak{T}(\mathcal{O})$ and call it *the deformation space of* $\mathbb{R}\mathrm{P}^2$-*structures* on $\mathcal{O}$. We remark that if $\mathcal{O}$ itself is a manifold , then hol_2 is a local homeomorphism. See Goldman [3] for more details.

4. Convex real projective structures

By an *affine patch* in the projective plane, we mean the complement of a projective line of $\mathbb{RP}^2$. An affine patch has a natural structure of an affine plane. Then a *convex set* in $\mathbb{RP}^2$ is an affinely convex subset of an affine patch in $\mathbb{RP}^2$. Now, as in the preceding section, let $\mathcal{O}$ be an orbifold $\widetilde{M}/\Gamma$. The deformation space $\mathfrak{C}(\mathcal{O})$ of *convex* real projective structures on $\mathcal{O}$ is the subspace of $\mathfrak{T}(\mathcal{O})$ consisting of equivalence classes of real projective structures each of which has a developing map $dev : \widetilde{M} \to \mathbb{RP}^2$ an embedding onto a convex subset of $\mathbb{RP}^2$. It is known that if $\mathcal{O}$ is a *closed orientable* surface of genus > 1, then the restriction to $\mathfrak{C}(\mathcal{O})$ of $hol : \mathfrak{T}(\mathcal{O}) \to \mathrm{Hom}(\Gamma, G)/G$ is an embedding onto a connected component of $\mathrm{Hom}(\Gamma, G)/G$. See Choi [1] for the proof. However for orbifolds, we do not have the proof.

5. The main part

We turn to our main discussion. Let $n_i \geq 2, i = 1, 2, 3, 4$, be integers such that $\sum(1/n_i) < 2$. There is a quadrilateral $\mathcal{P} = P_1P_2P_3P_4$ in $\mathbb{H}^2$ such that the angle at P_i is π/n_i for each i. Let Γ be the subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ generated by reflections in each side of $\mathcal{P}$. The group Γ acts properly discontinuously on $\mathbb{H}^2$ and $\mathcal{P}$ is a fundamental domain for Γ. Let us denote the reflection in the side P_iP_{i+1} by r_i. Then Γ admits a presentation

$$< r_1, r_2, r_3, r_4 | {r_i}^2 = (r_i r_{i+1})^{n_{i+1}} = 1, i = 1, 2, 3, 4 > \quad .$$

The quotient space $\mathcal{O} = \mathbb{H}^2/\Gamma$ is an orbifold of type $({}^*n_1n_2n_3n_4)$ and the quotient map $\mathbb{H}^2 \to \mathcal{O}$ is the universal covering. Throughout this section, n_i's are fixed and $\mathcal{O}$ will always mean the $({}^*n_1n_2n_3n_4)$ orbifold $\mathbb{H}^2/\Gamma$, where Γ is as above. The manifold $\mathbb{H}^2$ will be identified with the universal cover of $\mathcal{O}$ and $\mathfrak{D}', \mathfrak{D}, \mathfrak{T}$, and $\mathfrak{C}$ will be used in place of $\mathfrak{D}'(\mathcal{O}), \mathfrak{D}(\mathcal{O}), \mathfrak{T}(\mathcal{O})$, and $\mathfrak{C}(\mathcal{O})$, respectively. Our main purpose is to prove

THEOREM 1. *$\mathfrak{T}$ can be mapped continuously and bijectively onto the cell of dimension* $4 - |\{i | n_i = 2\}|$.

It is known that $\mathfrak{C} = \mathfrak{T}$; that is, each developing map $dev : \mathbb{H}^2 \to \mathbb{RP}^2$ is an embedding onto a convex set in $\mathbb{RP}^2$. Let $\mathfrak{H}$ be the image of *hol*. Then $\mathfrak{H}$ is the subspace of $\mathrm{Hom}(\Gamma, G)/G$ consisting of equivalence classes of homomorphisms each of which is the holonomy of an $\mathbb{RP}^2$-structure on $\mathcal{O}$. Fix a projective quadrilateral $\mathsf{p} = p_1p_2p_3p_4$ in $\mathbb{RP}^2$. (Let $p_1 = [0,0,1], p_2 = [1,0,1], p_3 = [1,1,1], p_4 = [0,1,1]$ in homogeneous coordinates for ease of computations.) Since for any two projective bases of $\mathbb{RP}^2$ there is a unique element in G carrying one to the other, we have a one-to-one correspondence between $\mathfrak{H}$ and $\mathfrak{h}$, where $\mathfrak{h}$ is the subset of $\mathrm{Hom}(\Gamma, G)$ consisting of holonomies corresponding to $dev \in \mathfrak{D}'$ such that $dev(P_i) = p_i$ for $i = 1,2,3,4$. Thus the restriction to $\mathfrak{h}$ of the quotient map $\mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G)/G$ is one-to-one and onto $\mathfrak{H}$.

PROOF OF THE THEOREM: We will show that $\mathfrak{h}$ is the cell of dimension $4 - \#\{i | n_i = 2\}$. It will follow easily from Lemmas 1 and 2.

LEMMA 1. *$H \in \mathrm{Hom}(\Gamma, G)$ is in $\mathfrak{h}$ if and only if*

$$H(r_1) = \begin{pmatrix} -1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & -b & -1 \end{pmatrix} \qquad H(r_2) = \begin{pmatrix} -c & 0 & c-1 \\ -d & -1 & d \\ -c-1 & 0 & c \end{pmatrix}$$

$$H(r_3) = \begin{pmatrix} -1 & f & -f \\ 0 & e & -e-1 \\ 0 & e-1 & -e \end{pmatrix} \qquad H(r_4) = \begin{pmatrix} -1 & 0 & 0 \\ -g & -1 & 0 \\ -h & 0 & 1 \end{pmatrix}$$

for $a, b, c, d, e, f, g, h \in \mathbb{R}$ satisfying

$$\begin{cases} d(a-b) = 2 + 2\cos(2\pi/n_2)\ ,\ d < 0 & \text{if } n_2 > 2 \\ d = 0\ ,\ a = b & \text{if } n_2 = 2 \end{cases} \qquad \text{(A)}$$

$$\begin{cases} (c-d+1)(-e+f+1) = 2 + 2\cos(2\pi/n_3)\ ,\ c-d+1 < 0 & \text{if } n_3 > 2 \\ c-d+1 = 0\ ,\ -e+f+1 = 0 & \text{if } n_3 = 2 \end{cases} \qquad \text{(B)}$$

$$\begin{cases} f(-g+h) = 2 + 2\cos(2\pi/n_4)\ ,\ f > 0 & \text{if } n_4 > 2 \\ f = 0\ ,\ g = h & \text{if } n_4 = 2 \end{cases} \qquad \text{(C)}$$

$$\begin{cases} ag = 2 + 2\cos(2\pi/n_1)\ ,\ a > 0 & \text{if } n_1 > 2 \\ a = 0\ ,\ g = 0 & \text{if } n_1 = 2 \end{cases} \qquad \text{(D)}$$

LEMMA 2. *The set of all* $(a,b,c,d,e,f,g,h) \in \mathbb{R}^8$ *satisfying* $(A),(B),(C)$ *and* (D) *is the cell of dimension* $4 - |\{i|n_i = 2\}|$.

PROOF OF LEMMA 1. We want to find all the holonomies corresponding to dev such that $dev(P_i) = p_i$. Let dev be such a developing map and H the holonomy corresponding to it. Let $A_i = H(r_i)$ for $i = 1,2,3,4$. From the equivariance relation $dev \circ \gamma = H(\gamma) \circ dev$ for each $\gamma \in \Gamma$ we see that A_i fixes the (projective) line p_ip_{i+1} pointwise since r_i fixes the (hyperbolic) line P_iP_{i+1} pointwise. So we get the following equations:

$$\left[A_1\begin{pmatrix} s \\ 0 \\ s+t \end{pmatrix}\right] = \begin{bmatrix} s \\ 0 \\ s+t \end{bmatrix} \qquad \forall s, t \in \mathbb{R} \tag{1}$$

$$\left[A_2\begin{pmatrix} s+t \\ t \\ s+t \end{pmatrix}\right] = \begin{bmatrix} s+t \\ t \\ s+t \end{bmatrix} \qquad \forall s, t \in \mathbb{R} \tag{2}$$

$$\left[A_3\begin{pmatrix} s \\ s+t \\ s+t \end{pmatrix}\right] = \begin{bmatrix} s \\ s+t \\ s+t \end{bmatrix} \qquad \forall s, t \in \mathbb{R} \tag{3}$$

$$\left[A_1\begin{pmatrix} 0 \\ s \\ s+t \end{pmatrix}\right] = \begin{bmatrix} 0 \\ s \\ s+t \end{bmatrix} \qquad \forall s, t \in \mathbb{R} \tag{4}$$

in the homogeneous coordinates.
From the relations ${r_i}^2 = 1$, we also have ${A_i}^2 = I$.

$$ {A_1}^2 = I \tag{5}$$

$$ {A_2}^2 = I \tag{6}$$

$$ {A_3}^2 = I \tag{7}$$

$$ {A_4}^2 = I \tag{8}$$

Note that $A_i \neq I$ from the equivariance.

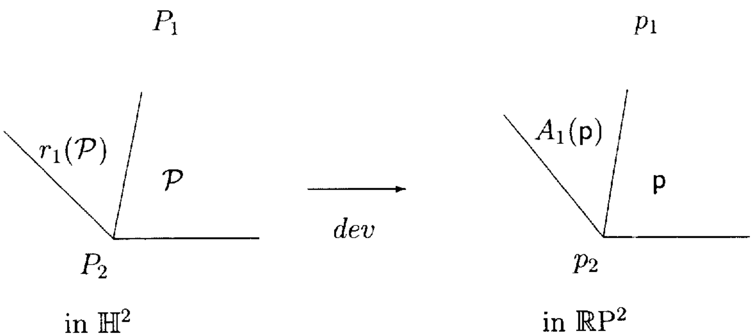

Figure 1: A figure illustrating the equivariance

So (1) and (5) mean that A_1 is a reflection in the line p_1p_2. Solving them, we get

$$A_1 = \begin{pmatrix} -1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & -b & -1 \end{pmatrix} \qquad a, b \in \mathbb{R} \tag{9}$$

Solving (2) and (6), we get

$$A_2 = \begin{pmatrix} -c & 0 & c-1 \\ -d & -1 & d \\ -c-1 & 0 & c \end{pmatrix} \qquad c, d \in \mathbb{R} \tag{10}$$

Solving (3) and (7), we get

$$A_3 = \begin{pmatrix} -1 & f & -f \\ 0 & e & -e-1 \\ 0 & e-1 & -e \end{pmatrix} \qquad e, f \in \mathbb{R} \tag{11}$$

Solving (4) and (8), we get

$$A_4 = \begin{pmatrix} -1 & 0 & 0 \\ -g & -1 & 0 \\ -h & 0 & 1 \end{pmatrix} \qquad g, h \in \mathbb{R} \tag{12}$$

Finally, we consider the relations $(r_i r_{i+1})^{n_{i+1}} = 1$ together with the equivariance. These induce the relations $(A_i A_{i+1})^{n_{i+1}} = I$. We will only consider the relation $(A_1A_2)^{n_2} = I$. Here we must take the two cases (a)$n_2 = 2$ and (b)$n_2 > 2$ separately.

(a) $n_2 = 2$: We have $(A_1A_2)^2 = I$ or $A_1A_2 = A_2A_1$. Substituting (9) and (10) into this equation, we get $d = 0, a = b$ from the following.

$$\begin{pmatrix} c+ad & a & -c-ad+1 \\ -d & -1 & -d \\ bd+c+1 & b & -bd-c \end{pmatrix} = \begin{pmatrix} c & ac-bc+b & -c+1 \\ d & ad-bd-1 & -d \\ c+1 & ac-bc+a & -c \end{pmatrix}$$

(b) $n_2 > 2$: Since A_1A_2 has 1 as an eigenvalue and $(A_1A_2)^{n_2} = I$, A_1A_2 has $e^{2\pi ki/n_2}$ and $e^{-2\pi ki/n_2}$ as the other two complex eigenvalues for a $k \in \{1, 2, \cdots, n_2 - 1\}$. Since r_1r_2 is the rotation around P_2 by the angle $2\pi/n_2$, A_1A_2 has $e^{2\pi i/n_2}$ and $e^{-2\pi i/n_2}$ as eigenvalues other than 1, by the equivariance. So the trace of A_1A_2 equals $1 + 2\cos(2\pi/n_2)$. Thus we get $d(a-b) = 2 + 2\cos(2\pi/n_2) \geq 0$. Note that for small positive values r,

$$\left[A_1\begin{pmatrix}1\\r\\1\end{pmatrix}\right] = \begin{bmatrix}-1-ar\\r\\-1-br\end{bmatrix} = \begin{bmatrix}\frac{1+ar}{1+br}\\\frac{-r}{1+br}\\1\end{bmatrix}$$

in the homogeneous coordinates.

We must have $d < 0$. Suppose on the contrary that $d > 0$ or, equivalently, $a > b$. Then the interiors of $A_1(\mathsf{p})$ and $A_2A_1(\mathsf{p})$ overlap. To see this, note that $(1+ar)/(1+br) > 1$ for small positive values r. So the intersection of $A_1(\mathsf{p})$ with an open neighborhood of p_2 looks like Fig. 2.

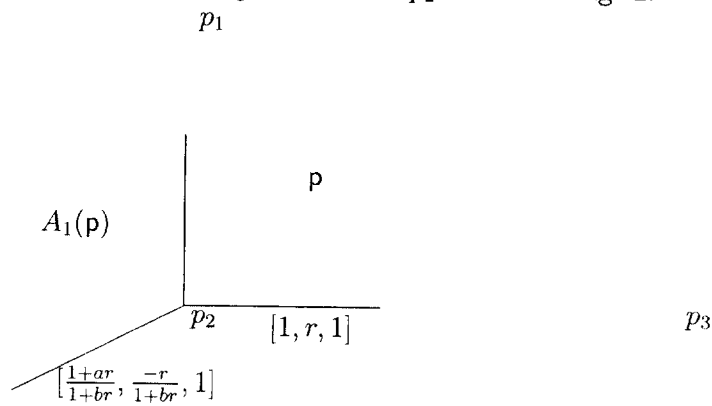

Figure 2

Let U be a small open neighborhood of p_2 such that *dev* restricted to an open neighborhood $\mathcal{U}$ of P_2 is a diffeomorphism onto U and let W be one of the four regions of U formed by the projective lines p_2p_1 and p_2p_3 in Fig. 3.

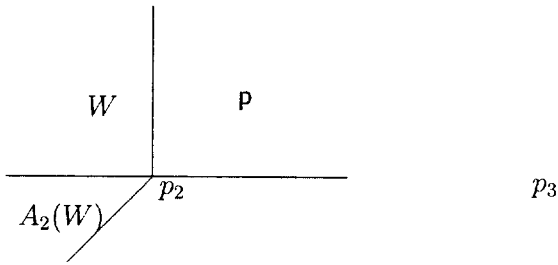

Figure 3

Taking sufficiently small U, we may assume that $W \subset A_1(\mathsf{p})$. Since A_2 is a reflection in p_2p_3, $A_2(W) \cap A_1(\mathsf{p})$ has nonempty interior. So $U \cap A_2A_1(\mathsf{p}) \cap A_1(\mathsf{p})$ has nonempty interior. Since $\mathcal{U} \cap r_2r_1(\mathcal{P}) \cap r_1(\mathcal{P})$ has the empty interior, *dev* is not a local diffeomorphism. Hence a contradiction. The other relations $(A_iA_{i+1})^{n_{i+1}} = I$, $i = 2, 3, 4$ can be treated in the same way.

Conversely, suppose $H \in \mathrm{Hom}(\Gamma, G)$ satisfies the conditions of Lemma 1. Then H induces a tessellation of a convex set $\Omega_H = \bigcup_{\gamma \in \Gamma} H(\gamma)(\mathsf{p})$. See Goldman [2] for the proof of the fact that Ω_H is convex. So it is evident that there is a $dev \in \mathfrak{D}'$ satisfying the equivariance relation : $dev \circ \gamma = H(\gamma) \circ dev \quad \forall \gamma \in \Gamma$. □

Proof of Lemma 2. We only consider the two cases (I) $n_1, n_2, n_3, n_4 > 2$ and (II) $n_1 = 2$, $n_2, n_3, n_4 > 2$. The other cases can be treated similarly.

(I) Let all n_i's be greater than 2. Define $p : \mathbb{R}^8 \to \mathbb{R}^4$ by

$$p(a, b, c, d, e, f, g, h) = (d, c - d + 1, f, a)$$

Let S be the set of all $(a, b, c, d, e, f, g, h) \in \mathbb{R}^8$ satisfying (A),(B),(C),(D). Then $p|_S$ is one-to-one and onto the subset $J = \{(x, y, z, w) \in \mathbb{R}^4 | x <$

$0, y < 0, z > 0, w > 0\}$ of $\mathbb{R}^4$: Define $j : J \to S$ by

$$\begin{aligned} j(x, y, z, w) = (&w, w - (2 + 2\cos(2\pi/n_2))/x, x + y - 1, x, \\ &z + 1 - (2 + 2\cos(2\pi/n_3))/y, z, (2 + 2\cos(2\pi/n_1))/w, \\ &(2 + 2\cos(2\pi/n_1))/w + (2 + 2\cos(2\pi/n_4))/z) \quad . \end{aligned}$$

This map was obtained just by solving (A), (B), (C), (D), letting $d = x, c - d + 1 = y, f = z, a = w$. So it is trivial that $p|_S : S \to J$ and j are inverses. Since J is a 4-cell, we are done in this case.

(II) Let $n_1 = 2,\ n_2, n_3, n_4 > 2$. Define $p : \mathbb{R}^8 \to \mathbb{R}^3$ by

$$p(a, b, c, d, e, f, g, h) = (d, c - d + 1, f)$$

Let S be as in Case(I) and $J = \{(x, y, z) \in \mathbb{R}^3 | x < 0, y < 0, z > 0\}$. Define $j : J \to S$ by

$$\begin{aligned} j(x, y, z) =(&0, -(2 + 2\cos(2\pi/n_2))/x, x + y - 1, x, z + 1, \\ &- (2 + 2\cos(2\pi/n_3))/y, z, 0, (2 + 2\cos(2\pi/n_4))/z) \quad . \end{aligned}$$

Then $p|_S : S \to J$ and j are inverses. Since J is a 3-cell, we are done in this case, too. □

References

[1] S. Choi and W. Goldman, *Convex real projective structures on closed surfaces are closed.*, Proc. Amer. Math. Soc., **118**, (1993), 657–661.

[2] W. Goldman, *Affine manifolds and projective geometry on surfaces*, Senior Thesis, Princeton Univ. (1977)

[3] W. Goldman, *Geometric structures and varieties of representations*, in "The Geometry of Group Representations", edited by W. Goldman and A. Magid, Contemp. Math., **74**, Amer. Math. Soc., Providence, R.I. (1988), 169–198.

[4] M. Hirsch, *Differential Topology*, Graduate Texts in Math., **33**, Springer-Verlag, New York (1976)

[5] J. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts in Math., **149**, Springer-Verlag, New York (1994)

[6] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc., **15** (1984), 401– 487

[7] W. Thurston, *The Geometry and Topology of 3-manifolds*, lecture notes, Princeton Univ., Princeton, N. J. (1990)

Department of Mathematics, College of Natural Sciences, Seoul National University, Seoul 151-742, Korea

E-mail: jklee@math.snu.ac.kr