Bull. Korean Math. Soc. **34** (1997), No. 4, pp. 617–626

ON THE STRONG LAW OF LARGE NUMBERS FOR PAIRWISE I. I. D. RANDOM VARIABLES

SOO HAK SUNG

ABSTRACT. This paper is concerned with the general strong law of large numbers for pairwise independent identically distributed random variables. Necessary and sufficient conditions for the SLLN are obtained.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables, and put $S_n = \sum_{i=1}^{n} X_i$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants such that $0 < b_n \uparrow \infty$. Then $\{X_n, n \geq 1\}$ is said to obey the general strong law of large numbers(SLLN) with centering constants $\{a_n, n \geq 1\}$ and norming constants $\{b_n, n \geq 1\}$ if

$$\frac{S_n - a_n}{b_n} \to 0 \text{ almost surely.} \tag{1}$$

There are two famous SLLNs for a sequence $\{X_n, n \geq 1\}$ of independent identically distributed random variables: Kolmogorov's SLLN and Marcinkiewicz-Zygmund's SLLN. Kolmogorov's SLLN states that $E|X_1| < \infty$ and $EX_1 = c$ if and only if (1) holds with $a_n = cn$ and $b_n = n$. Marcinkiewicz-Zygmund's SLLN states that, for $1 < r < 2$, $E|X_1|^r < \infty$ and $EX_1 = c$ if and only if (1) holds with $a_n = cn$ and $b_n = n^{1/r}$.

Received February 11, 1997.

1991 Mathematics Subject Classification: 60F15.

Key words and phrases: Strong law of large numbers, pairwise independent random variables.

This paper was supported by the Basic Science Research Institute Program, Ministry of Education, the Republic of Korea, 1996, Project No. BSRI-96-1439.

Now let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent identically distributed(pairwise i.i.d.) random variables. Etemadi([5], [6])proved Kolmogorov's SLLN for pairwise i.i.d. random variables. Recently, Martikainen [9] showed that, for $1 < r < 2$, the condition $E|X_1|^r(\log^+ |X|)^\tau < \infty$(for some positive $\tau > 4r - 6$) is sufficient for the relation $(S_n - ES_n)/n^{1/r} \to 0$ almost surely. However, it is not known yet that Marcinkiewicz-Zygmund's SLLN holds for pairwise i.i.d. random variables. Extending certain idea of Etemadi, many authors studied SLLN for pairwise independent random variables; see Chandra and Goswami [1], Csorgo et al([3], [4]), and Etemadi [7].

In this paper, we examine the connection between moment condition and SLLN to a sequence of pairwise i.i.d. random variables.

Throughout this paper, C denotes a positive constant which may be different in various places, and $I(A)$ the indicator function of event A. Also $f(x)$ is a non-decreasing function on $[0, \infty)$ such that $f(x) > 0$ and $f(x) \uparrow \infty$ as $x \uparrow \infty$.

2. Main Result

First we get a SLLN for pairwise independent, but not necessarily identically distributed, random variables. To do this, we need the next lemma which is well known(see, Loeve [8], p. 124).

LEMMA 1. *Let $\{X_n, n \geq 1\}$ be a sequence of orthogonal random variables. If $\sum_{n=1}^{\infty} \log^2 n E X_n^2 < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges almost surely.*

THEOREM 1. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent random variables and let $\{b_n, n \geq 1\}$ be a sequence of constants satisfying $0 < b_n \uparrow \infty$. Assume that*

(i) $\sum_{n=1}^{\infty} P(|X_n| > b_n) < \infty$;
(ii) $\sum_{n=1}^{\infty} E X_n^2 I(|X_n| \leq b_n)/b_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \log^2 n E X_n^2 I(|X_n| \leq b_n/\log^2 n)/b_n^2 < \infty$;
and
(iv) $\sum_{i=1}^{n} E|X_i| I(|X_i| > b_i/\log^2 i)/b_n \to 0$.

Then $(S_n - ES_n)/b_n \to 0$ almost surely.

Proof. Let $X'_n = X_n I(|X_n| \le b_n/\log^2 n), X''_n = X_n I(|X_n| > b_n), X'''_n = X_n - X'_n - X''_n, n \ge 1$. Then (i) implies by the Borel-Cantelli lemma that

$$\frac{\sum_{i=1}^n X''_i}{b_n} \to 0 \text{ almost surely.} \tag{2}$$

Since $\{X_n\}$ is pairwise independent random variables, $\{(X'_n - EX'_n)/b_n\}$ is orthogonal random variables. Hence (iii) entails by Lemma 1 that $\sum_{n=1}^\infty (X'_n - EX'_n)/b_n$ converges almost surely. From Kronecker lemma we see that

$$\frac{\sum_{i=1}^n (X'_i - EX'_i)}{b_n} \to 0 \text{ almost surely.} \tag{3}$$

Combining (2), (3), and (iv) yields

$$\frac{\sum_{i=1}^n (X'_i + X''_i - EX_i)}{b_n} \to 0 \text{ almost surely.}$$

Hence, it is enough to show that

$$\frac{\sum_{i=1}^n X'''_i}{b_n} \to 0 \text{ almost surely.} \tag{4}$$

To prove (4), for $k \ge 1$ we define $m_k = \inf\{n : b_n \ge 2^k\}$. Then for $m_k \le n < m_{k+1}$

$$\frac{|\sum_{i=1}^n X'''_i|}{b_n} \le \frac{\sum_{i=1}^{m_{k+1}-1} (|X'''_i| - E|X'''_i|)}{b_{m_k}} + \frac{\sum_{i=1}^{m_{k+1}-1} E|X'''_i|}{b_{m_k}}. \tag{5}$$

The second term on the right-hand of (5) converges to 0 by (iv), since

$$\begin{aligned}\frac{\sum_{i=1}^{m_{k+1}-1} E|X'''_i|}{b_{m_k}} &\le \frac{\sum_{i=1}^{m_{k+1}-1} E|X_i|I(|X_i| > b_i/\log^2 i)}{b_{m_k}} \\ &\le \frac{2\ \sum_{i=1}^{m_{k+1}-1} E|X_i|I(|X_i| > b_i/\log^2 i)}{b_{m_{k+1}-1}} \to 0.\end{aligned}$$

Using Markov's inequality, the pairwise independence of $\{X_n\}$, and (ii), we have that for every $\epsilon > 0$

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P\left(\frac{\mid \sum_{i=1}^{m_{k+1}-1}(|X_i'''| - E|X_i'''|) \mid}{b_{m_k}} > \epsilon\right) \\
\leq & \frac{1}{\epsilon^2} \sum_{k=1}^{\infty} \frac{1}{b_{m_k}^2} \sum_{i=1}^{m_{k+1}-1} E|X_i'''|^2 \\
\leq & \frac{1}{\epsilon^2} \sum_{k=1}^{\infty} \frac{1}{b_{m_k}^2} \sum_{i=1}^{m_{k+1}-1} EX_i^2 I(|X_i| \leq b_i) \\
= & \frac{1}{\epsilon^2} \sum_{i=1}^{\infty} EX_i^2 I(|X_i| \leq b_i) \sum_{\{k: m_{k+1}-1 \geq i\}} \frac{1}{b_{m_k}^2} \\
\leq & C \sum_{i=1}^{\infty} \frac{EX_i^2 I(|X_i| \leq b_i)}{b_i^2} < \infty.
\end{aligned}
$$

The third inequality follows from the following:

$$
\begin{aligned}
\sum_{\{k: m_{k+1}-1 \geq i\}} \frac{1}{b_{m_k}^2} &= \sum_{k=k_0}^{\infty} \frac{1}{b_{m_k}^2} \leq \sum_{k=k_0}^{\infty} \frac{1}{2^{2k}} \\
&= \frac{16}{3} \frac{1}{2^{2(k_0+1)}} < \frac{16}{3 b_{m_{k_0+1}-1}^2} \leq \frac{16}{3 b_i^2},
\end{aligned}
$$

where $k_0 = \min\{k : m_{k+1} - 1 \geq i\}$. Thus it follows from the Borel-Cantelli lemma that the first term on the right-hand of (5) converges to 0 almost surely, and so (4) holds. □

From now on $\{X_n, n \geq 1\}$ will denote a sequence of pairwise i.i.d. random variables. To find necessary and sufficient conditions for SLLN, we need the following several lemmas. Lemma 2 is a generalization of Lemma 3.3.2 in Stout [10]. Lemma 3 is well known(see, Chow and Teicher [2], p. 116-117).

LEMMA 2. *Let X be a random variable and let $\{b_n, n \geq 1\}$ be a sequence of constants satisfying $0 < b_n \uparrow \infty$. Let ϕ be any non-decreasing function on $[0, \infty)$ such that for some constants $C_1 > 0$ and $C_2 > 0$*

$$C_1 \leq \frac{\phi(b_n)}{n} \leq C_2. \tag{6}$$

Then the following two conditions are equivalent:

(i) $\sum_{n=1}^{\infty} P(|X| > b_n) < \infty$;
(ii) $E[\phi(|X|)] < \infty$.

Proof. (i) $\Rightarrow$ (ii). Since $\phi(x)$ is non-decreasing, we have

$$\begin{aligned} E[\phi(|X|)] &= \sum_{n=1}^{\infty} E[\phi(|X|)I(b_{n-1} < |X| \leq b_n)] \quad (b_0 \equiv 0) \\ &\leq \sum_{n=1}^{\infty} \phi(b_n)P(b_{n-1} < |X| \leq b_n) \\ &\leq C_2 \sum_{n=1}^{\infty} nP(b_{n-1} < |X| \leq b_n) \\ &= C_2\{1 + \sum_{n=1}^{\infty} P(|X| > b_n)\} < \infty. \end{aligned}$$

The proof of (ii) $\Rightarrow$ (i) is similar to that of (i) $\Rightarrow$ (ii), and omitted. □

LEMMA 3. *Let $\{b_n, n \geq 1\}$ be a sequence of constants such that $0 < b_n \uparrow \infty$ and $b_n^2 \sum_{i=n}^{\infty} 1/b_i^2 = O(n)$. If $\sum_{n=1}^{\infty} P(|X| > b_n) < \infty$, then*

$$\sum_{n=1}^{\infty} \frac{EX^2 I(|X| \leq b_n)}{b_n^2} < \infty.$$

The next lemma gives a sufficient condition to guarantee (iii) of Theorem 1 when $\{X_n, n \geq 1\}$ is a sequence of pairwise i.i.d. random variables. Recall that $f(x)$ is a non-decreasing function on $[0, \infty)$ such that $f(x) > 0$ and $f(x) \uparrow \infty$ as $x \uparrow \infty$.

LEMMA 4. *Let X be a random variable and let $\{b_n, n \geq 1\}$ be a sequence of constants such that*

$$\frac{b_n}{\log^2 n} \uparrow \text{ and } \frac{b_n^2}{\log^2 n} \sum_{i=n}^{\infty} \frac{\log^2 i}{b_i^2} = O(n). \tag{7}$$

Assume that

$$\frac{x^2}{f(x)} \uparrow \quad \text{and} \quad \frac{b_n^2}{f(b_n/\log^2 n) n \log^2 n} \geq C \tag{8}$$

for some constant $C > 0$. If $E[X^2/f(|X|)] < \infty$, then

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{b_n^2} E[X^2 I(|X| \leq \frac{b_n}{\log^2 n})] < \infty.$$

Proof. By the second half of (8) we have $n/\log^2 n \leq b_n^2/\{Cf(b_n/\log^2 n)\log^4 n\}$. From this result, (7), and the first half of (8), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\log^2 n}{b_n^2} E[X^2 I(|X| \leq \frac{b_n}{\log^2 n})] \\
=&\sum_{n=1}^{\infty} \frac{\log^2 n}{b_n^2} \sum_{i=1}^{n} E[X^2 I(\frac{b_{i-1}}{\log^2(i-1)} < |X| \leq \frac{b_i}{\log^2 i})] \quad (b_0 \equiv 0, \log 0 \equiv 1) \\
=&\sum_{i=1}^{\infty} E[X^2 I(\frac{b_{i-1}}{\log^2(i-1)} < |X| \leq \frac{b_i}{\log^2 i})] \sum_{n=i}^{\infty} \frac{\log^2 n}{b_n^2} \\
\leq& C\sum_{i=1}^{\infty} E[X^2 I(\frac{b_{i-1}}{\log^2(i-1)} < |X| \leq \frac{b_i}{\log^2 i})] \frac{i \log^2 i}{b_i^2} \\
\leq& C\sum_{i=1}^{\infty} P(\frac{b_{i-1}}{\log^2(i-1)} < |X| \leq \frac{b_i}{\log^2 i}) \frac{i}{\log^2 i} \\
\leq& C\sum_{i=1}^{\infty} P(\frac{b_{i-1}}{\log^2(i-1)} < |X| \leq \frac{b_i}{\log^2 i}) \frac{i-1}{\log^2(i-1)} \\
\leq& C\sum_{i=1}^{\infty} P(\frac{b_{i-1}}{\log^2(i-1)} < |X| \leq \frac{b_i}{\log^2 i}) \frac{b_{i-1}^2}{f(b_{i-1}/\log^2(i-1))\log^4(i-1)} \\
\leq& CE[\frac{X^2}{f(|X|)}].
\end{aligned}$$

Note that the last inequality follows from $x^2/f(x) \uparrow$. Thus the proof is completed. $\square$

The following lemma gives a sufficient condition to guarantee (iv) of Theorem 1 when $\{X_n, n \geq 1\}$ is a sequence of pairwise i.i.d. random variables.

LEMMA 5. *Let X be a random variable and let $\{b_n, n \geq 1\}$ be a sequence of constants such that*

$$\frac{b_n}{\log^2 n} \uparrow \infty \quad \text{and} \quad \frac{1}{b_n}\sum_{i=1}^{n}\frac{b_i}{i} = O(1). \tag{9}$$

Assume that $f(n) = b_n$ for all $n \geq 1$,

$$\frac{x}{f(x)} \uparrow \quad \text{and} \quad \frac{b_n^2}{f(b_n/\log^2 n)n\log^2 n} \geq C \tag{10}$$

for some constant $C > 0$. If $E[X^2/f(|X|)] < \infty$, then

$$\frac{\sum_{i=1}^{n} E|X|I(|X| > b_i/\log^2 i)}{b_n} \to 0.$$

Proof. First we show that $E|X| < \infty$. Since $x/f(x) \uparrow$, it follows that

$$\begin{aligned} E[\frac{X^2}{f(|X|)}] &= E[\frac{X^2}{f(|X|)}(I(|X| \leq a) + I(|X| > a))] \\ &\geq E[\frac{X^2}{f(|X|)}I(|X| > a)] \\ &\geq \frac{a}{f(a)}E[|X|I(|X| > a)]. \end{aligned}$$

Hence we obtain

$$\begin{aligned} E|X| &= E|X|I(|X| \leq a) + E|X|I(|X| > a) \\ &\leq a + \frac{f(a)}{a}E[\frac{X^2}{f(|X|)}] < \infty. \end{aligned}$$

Now we let $d_n = [\log b_n]$, where $[A]$ is the integer part of A. By the second part of (10), we have $n/b_n \le b_n/\{Cf(b_n/\log^2 n)\log^2 n\}$. From this result, (9), and the first part of (10), we have

$$
\begin{aligned}
&\frac{\sum_{i=1}^{n} E|X|I(|X| > b_i/\log^2 i)}{b_n} \\
=&\frac{\sum_{i=1}^{d_n} E|X|I(|X| > b_i/\log^2 i)}{b_n} + \frac{\sum_{i=d_n+1}^{n} E|X|I(|X| > b_i/\log^2 i)}{b_n} \\
\le&\frac{d_n E|X|}{b_n} + \frac{1}{b_n}\sum_{i=d_n+1}^{n}\frac{b_i}{i}\frac{b_i}{Cf(b_i/\log^2 i)\log^2 i}E|X|I(|X| > \frac{b_i}{\log^2 i}) \\
\le&\frac{d_n E|X|}{b_n} + \frac{1}{Cb_n}\sum_{i=d_n+1}^{n}\frac{b_i}{i}E[\frac{X^2}{f(|X|)}I(|X| > \frac{b_i}{\log^2 i})] \\
\le&\frac{d_n E|X|}{b_n} + \frac{1}{C}E[\frac{X^2}{f(|X|)}I(|X| > \frac{f(d_n)}{\log^2 d_n})]\frac{\sum_{i=1}^{n} b_i/i}{b_n} \\
\le&\frac{d_n E|X|}{b_n} + CE[\frac{X^2}{f(|X|)}I(|X| > \frac{f(d_n)}{\log^2 d_n})] \to 0,
\end{aligned}
$$

since $d_n/b_n \to 0$ and $f(d_n)/\log^2 d_n \uparrow \infty$. □

Now we state and prove our main theorem.

THEOREM 2. *Let $\{X_n, n \ge 1\}$ be a sequence of pairwise i.i.d. random variables and let $b_n = f(n)$ for all $n \ge 1$. Assume that*

(a) $x/f(x) \uparrow$;
(b) $f(x)/\log^2 x \uparrow \infty$;
(c) $b_n^2 \sum_{i=n}^{\infty} 1/b_i^2 = O(n)$;
(d) $b_n^2(\sum_{i=n}^{\infty} \log^2 i/b_i^2)/\log^2 n = O(n)$;
(e) $(\sum_{i=1}^{n} b_i/i)/b_n = O(1)$;
(f) $C_1 \le b_n^2/\{nf(b_n)\} \le C_2$ *for some constants* $C_1 > 0$ *and* $C_2 > 0$; *and*
(g) $b_n^2/\{f(b_n/\log^2 n)n\log^2 n\} \ge C_3$ *for some constant* $C_3 > 0$.

Then $E[X_1^2/f(|X_1|)] < \infty$ *if and only if* $(S_n - ES_n)/b_n \to 0$ *almost surely.*

Proof. Note that (a) and (f) imply by Lemma 2 that

$$(11) \qquad E[\frac{X_1^2}{f(|X_1|)}] < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} P(|X_1| > b_n) < \infty.$$

The necessity is proved by showing that the conditions of Theorem 1 are satisfied. Assume that $E[X_1^2/f(|X_1|)] < \infty$. Then condition (i) of Theorem 1 holds by (11). Conditions (ii), (iii), and (iv) hold by Lemma 3, Lemma 4, and Lemma 5, respectively.

Conversely, assume that $(S_n - ES_n)/b_n \to 0$ almost surely. Since $0 < b_n \uparrow \infty$, it follows that

$$\frac{|X_n|}{b_n} \le \frac{|S_n - ES_n|}{b_n} + \frac{|S_{n-1} - ES_{n-1}|}{b_{n-1}} + \frac{|EX_1|}{b_n} \to 0 \text{ almost surely.}$$

Hence we have by the second Borel-Cantelli lemma that

$$\sum_{n=1}^{\infty} P(|X_1| > b_n) < \infty,$$

which is equivalent to $E[X_1^2/f(|X_1|)] < \infty$ by (11). Thus the sufficiency is proved. □

REMARK 1. Theorem 2 does not imply Marcinkiewicz-Zygmund's SLLN.

Finally we consider corollaries to Theorem 2 that correspond to some functions $f(x)$. Since $f(x) = x/\exp\{\log^{\alpha} x\}(0 \le \alpha < 1)$ or $x/\log^{\beta} x(\beta \ge 0)$ satisfies the conditions (a)-(g) of Theorem 2, we have the following corollaries.

COROLLARY 1. *Let $\{X_n, n \ge 1\}$ be a sequence of pairwise i.i.d. random variables with $E[|X_1|\exp\{(\log^+ |X_1|)^{\alpha}\}] < \infty$ for some $0 \le \alpha < 1$. Then*

$$\frac{S_n - ES_n}{n/\exp\{(\log n)^{\alpha}\}} \to 0 \textit{ almost surely.}$$

COROLLARY 2. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise i.i.d. random variables with $E[|X_1|(\log^+ |X_1|)^\beta] < \infty$ for some $\beta \geq 0$. Then*

$$\frac{S_n - ES_n}{n/(\log n)^\beta} \to 0 \text{ almost surely.}$$

REMARK 2. Corollary 1 reduces to Etemadi's SLLN when $\alpha = 0$. Also Corollary 2 implies Etemadi's SLLN when $\beta = 0$.

References

[1] T. K. Chandra and A. Goswami, *Cesaro uniform integrability and the strong law of large numbers*, Sankhya Series A **54** (1992), 215–231.

[2] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Intechangeability, Martingales*, Springer-Verlag, New York -Heidelberg-Berlin, 1978.

[3] S. Csorgo, K. Tandori, and V. Totik, *On the strong law of large numbers for pairwise independent random variables*, Acta Math. Hungar. **42** (1983), 319–330.

[4] ———, *On the convergence of series of pairwise independent random variables*, Acta Math. Hungar. **45** (1985), 445–450.

[5] N. Etemadi, *An elementary proof of the strong law of large numbers*, Z. Wahrsch. Verw. Gebiete **55** (1981), 119–122.

[6] ———, *On the laws of large numbers for nonnegative random variables*, J. Multivariate Analysis **13** (1983), 187–193.

[7] ———, *Stability of sums of weighted nonnegative random variables*, J. Multivariate Analysis **13** (1983), 361–365.

[8] M. Loeve, *Probability Theory* II, 4th ed., Springer-Verlag, 1977.

[9] A. Martikainen, *On the strong law of large numbers for sums of pairwise independent random variables*, Statist & Probability Letters **25** (1995), 21–26.

[10] W. F. Stout, *Almost Sure Convergence*, Academic Press, New York, 1974.

DEPARTMENT OF APPLIED MATHEMATICS, PAI CHAI UNIVERSITY, TAEJON 302-735, KOREA

E-mail: sungsh@woonam.paichai.ac.kr