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INTEGRAL OPERATORS THAT PRESERVE THE SUBORDINATION

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ABSTRACT. Let $H(U)$ be the space of all analytic functions in the unit disk U and let $\mathcal{K} \subset H(U)$. For the operator $A_{\beta,\gamma} : \mathcal{K} \longrightarrow H(U)$ defined by

$$A_{\beta,\gamma}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}\,dt\right]^{1/\beta}$$

and $\beta, \gamma \in \mathbf{C}$, we determined conditions on $g(z), \beta$ and γ such that

$$z\left[\frac{f(z)}{z}\right]^\beta \prec z\left[\frac{g(z)}{z}\right]^\beta \quad \text{implies} \quad z\left[\frac{A_{\beta,\gamma}(f)(z)}{z}\right]^\beta \prec z\left[\frac{A_{\beta,\gamma}(g)(z)}{z}\right]^\beta$$

and we presented some particular cases of our main result.

1. Introduction

Let $H(U)$ be the space of all analytic functions in the unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$ and let $f, g \in H(U)$. We say that f is *subordinate* to g , written $f(z) \prec g(z)$, if g is univalent in U, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

In [7] the authors determined conditions under which

$$f(z) \prec g(z) \quad \text{implies} \quad A(f)(z) \prec A(g)(z)$$

where $A : K \longrightarrow H(U)$, $K \subset H(U)$ and $A(f)(z) = \left[\frac{1}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}\,dt\right]^{1/\beta}$, $\beta, \gamma \in \mathbf{C}$.

Note that some particular cases of this result were previously obtained in [2], [3] and [9].

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For $h \in \mathcal{A}$, $\mathcal{A} \subset H(U)$, considering the integral operator $A_h : \widetilde{K} \longrightarrow H(U)$, $\widetilde{K} \subset H(U)$ defined by

$$A_h(f)(z) = \left[\beta \int_0^z f^\beta(t)h^{-1}(t)h'(t)\,dt\right]^{1/\beta}, \beta \in \mathbf{C}$$

in [1] the author gives sufficient conditions on $h(z)$ and $g(z)$ such that

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \quad \text{implies} \quad A_h(f)(z) \prec A_h(g)(z).$$

Let the integral operator $A_{\beta,\gamma} : \mathcal{K} \longrightarrow H(U)$, $\mathcal{K} \subset H(U)$ defined by

$$(1) \qquad A_{\beta,\gamma}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma} \int_0^z f^\beta(t)t^{\gamma-1}(t)\,dt\right]^{1/\beta}, \quad \beta, \gamma \in \mathbf{C}.$$

In the present paper we determine conditions on $g(z), \beta$ and γ so that the next implication holds :

$$z\left[\frac{f(z)}{z}\right]^\beta \prec z\left[\frac{g(z)}{z}\right]^\beta \Longrightarrow z\left[\frac{A_{\beta,\gamma}(f)(z)}{z}\right]^\beta \prec z\left[\frac{A_{\beta,\gamma}(g)(z)}{z}\right]^\beta$$

and in addition some particular cases obtained for different choices of β, γ and $g(z)$ will be given.

2. Preliminaries

In order to prove our main results, we will need the next definitions and lemmas presented in this section.

Let $c \in \mathbf{C}$ with $Re\ c > 0$ and let $N = N(c) = \dfrac{|c|\sqrt{1 + 2Re\ c} + Im\ c}{Re\ c}$. If k is the univalent function $k(z) = \dfrac{2Nz}{1-z^2}$ then we define the *"open door"* function R_c by

$$(2) \qquad R_c(z) = k\left(\frac{z+b}{1+\bar{b}z}\right), \ z \in U.$$

Note that R_c is univalent in U, $R_c(0) = c$ and $R_c(U) = k(U)$ is the complex plane slit along the half-lines $Re\ w = 0$, $Im\ w \geq N$ and $Re\ w = 0$, $Im\ w \leq -N$.

Let A be the set of functions $f(z) = z + a_2 z^2 + \cdots$ that are analytic in the unit disk U and we denote by $D = \{\phi \in H(U) : \phi(z) \neq 0 \text{ for } z \in U, \phi(0) = 1\}$.

LEMMA 2.1. [6] *Let* $\phi, \Phi \in D$ *and let* $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ *with* $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ *and* $Re\ (\alpha + \delta) > 0$. *If* $f \in A$ *satisfies*

$$\alpha \frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{\alpha+\delta}(z),$$

where R_c *is defined by* (2) *and if the function* F *is defined by*

$$F = A_{\beta\ ,\gamma}(f) \tag{3}$$

then

$$F \in A, \ \frac{F(z)}{z} \neq 0, \ z \in U \text{ and } Re \left[\beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0, \ z \in U.$$

(*All powers in* (1) *are principal ones.*)

A function $L(z;t)$, $z \in U$, $t \geq 0$ is called to be *a subordination* (*or a Loewner*) *chain* if $L(\cdot;t)$ is analytic and univalent in U for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0, +\infty)$ for all $z \in U$ and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$.

LEMMA 2.2. [8, p. 159] *The function* $L(z;t) = a_1(t)z + \cdots$ *with* $a_1(t) \neq 0$ *for all* $t \geq 0$ *and* $\lim_{t \to +\infty} |a_1(t)| = +\infty$ *is a subordination chain if and only if*

$$Re \left[z \frac{\partial L / \partial z}{\partial L / \partial t}\right] > 0, \ z \in U, \ t \geq 0.$$

A function $f \in A$ is called to be a *convex* (*and univalent*) *function* in U if $Re \left[1 + \frac{zf''(z)}{f'(z)}\right] > 0$, $z \in U$ and we represent the class of such functions by K . We denote by $K(\gamma)$, $\gamma \leq 1$ the class of *convex functions if order* γ , i.e.

$$K(\gamma) = \left\{ f \in A : Re \left[1 + \frac{zf''(z)}{f'(z)}\right]\right\} > \gamma, \ z \in U.$$

LEMMA 2.3. [5] *Let* F *be analytic in* U *and let* G *be analytic and univalent in* $\bar{U}$ *with* $F(0) = G(0)$. *If* F *is not subordinate to* G , *the there exist points* $z_0 \in U$, $\zeta_0 \in \partial U$ *an* $m \geq 1$ *for which* $F(|z| < |z_0|) \subset G(U)$, $F(z_0) = G(z_0)$ *and* $z_0 F'(z_0) = m\zeta_0 G'(\zeta_0)$.

LEMMA 2.4. *Suppose that the function* $\psi : \mathbf{C}^2 \times U \longrightarrow \mathbf{C}$ *satisfies the condition* $Re\ \psi(is, t; z) \leq 0$ *for all* $s \in R,\ t \leq -\frac{1+s^2}{2}$ *and all* $z \in U$. *If* $p \in H(U)$ *with* $p(0) = 1$ *then*

$$Re\ \psi(p(z), zp'(z); z) > 0,\ z \in U \text{ implies } Re\ p(z) > 0,\ z \in U.$$

More general forms of this lemma may be found in [5].

LEMMA 2.5. [5] *Let* $\beta, \gamma \in \mathbf{C}$ *with* $\beta \neq 0$ *and let* $h \in H(U)$ *with* $h(0) = c$. *If* $Re\ [\beta h(z) + \gamma] > 0,\ z \in U$ *then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z)$$

with $q(0) = c$ *is regular in* U *and satisfies* $Re\ [\beta q(z) + \gamma] > 0,\ z \in U$.

Finally we denote by $\mathcal{F}_{\beta,\gamma}$ the class of functions $f \in A$ that satisfy

$$\beta \frac{zf'(z)}{f(z)} + \gamma \prec R_{\beta+\gamma}(z).$$

3. Main Results

First we will determine the subset, $\mathcal{K} \subset H(U)$ such that the integral operator given by (1) will be well defined, considering a more general form of this operator.

LEMMA 3.1. *Let* $\beta, \gamma \in \mathbf{C}$ *with* $\beta \neq 0,\ Re\ (\beta + \gamma) > 0$ *and let* $h \in A$ *with* $h(z)h'(z)/z \neq 0,\ z \in U$. *If* $f \in A$ *and*

$$\beta \frac{zf'(z)}{f(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)} \prec R_{\beta+\gamma}(z)$$

then

$$F \in A,\ \frac{F(z)}{z} \neq 0,\ z \in U \text{ and } Re\ \left[\beta \frac{zF'(z)}{F(z)} + \gamma \frac{zh'(z)}{z}\right] > 0,\ z \in U$$

where

$$F(z) = I_h(z) = \left[\frac{\beta + \gamma}{h^\gamma(z)} \int_0^z f^\beta(t) h^{\gamma-1}(t) h'(t)\ dt\right]^{1/\beta}.$$

Proof. In order to prove the above result we will use Lemma 2.1 for $\alpha = \beta$, $\Phi(z) = [h(z)/z]^{\gamma}$ and $\phi(z) = [h(z)/z]^{\gamma-1}h'(z)$. From the assumptions we have $\Phi, \phi \in D$, $A_{\beta,\gamma} = I_h$ and a simple calculus shows that the conditions of Lemma 2.1 are satisfied , hence we obtain our result. □

REMARK. Taking $h(z) = z$ in Lemma 3.1 and using the fact that $I_h = A_{\beta,\gamma}$, for the case $h(z) = z$ we have the next implication :

Let $\beta, \gamma \in \mathbf{C}$ with $\beta \neq 0$, $Re\ (\beta + \gamma) > 0$. Then $f \in \mathcal{F}_{\beta,\gamma}$ implies $F \in A$, $\frac{F(z)}{z} \neq 0$, $z \in U$ and $Re\ \left[\beta \frac{zF'(z)}{F(z)} + \gamma\right] > 0$, $z \in U$ where $F(z) = A_{\beta,\gamma}(f)(z)$.

THEOREM 1. *Let $\beta, \gamma \in \mathbf{C}$ with $\beta \neq 0$, $0 < \beta + \gamma \leq 1$. Let $f, g \in \mathcal{F}_{\beta,\gamma}$ and for $\beta \neq 1$ suppose in addition that $f(z)/z \neq 0$, $g(z)/z \neq 0$, $z \in U$.*

$$If \quad Re\left[1 + \frac{z\phi''(z)}{\phi'(z)}\right] > 1 - (\beta + \gamma),\ where\ \phi(z) = z\left[\frac{g(z)}{z}\right]^{\beta}\ then$$

$$z\left[\frac{f(z)}{z}\right]^{\beta} \prec z\left[\frac{g(z)}{z}\right]^{\beta}\ implies\ z\left[\frac{A_{\beta,\gamma}(f)(z)}{z}\right]^{\beta} \prec z\left[\frac{A_{\beta,\gamma}(g)(z)}{z}\right]^{\beta}.$$

Proof. Denoting $F = A_{\beta,\gamma}(f)$, $G = A_{\beta,\gamma}(g)$, $\psi(z) = z[f(z)/z]^{\beta}$, $\phi(z) = z[g(z)/z]^{\beta}$, $\Psi(z) = z[F(z)/z]^{\beta}$, $\Phi(z) = z[G(z)/z]^{\beta}$, we need to prove that $\psi(z) \prec \phi(z)$ implies $\Psi(z) \prec \Phi(z)$. Then $\psi, \phi \in A$ and by the above remark we have $F(z)/z \neq 0$ and $G(z)/z \neq 0$ hence $\Psi, \Phi \in H(U)$ and moreover $\Psi, \Phi \in A$.

Differentiating the equality $G(z) = A_{\beta,\gamma}(g)(z)$ we have

$$G^{\beta}(z)\left[\beta\frac{zG'(z)}{G(z)} + \gamma\right]\frac{1}{\beta + \gamma} = g^{\beta}(z). \tag{4}$$

Since $\Phi(z) = z\left[\frac{G(z)}{z}\right]^{\beta}$, by differentiating this relation we obtain

$$\beta\frac{zG'(z)}{G(z)} + \gamma = \beta + \gamma - 1 + \frac{z\Phi'(z)}{\Phi(z)}$$

and replacing this in (4) we deduce that

$$\phi(z) = \left(1 - \frac{1}{\beta + \gamma}\right)\Phi(z) + \frac{1}{\beta + \gamma}z\Phi'(z). \tag{5}$$

Letting $L(z;t)=\left(1-\frac{1}{\beta+\gamma}\right)\Phi(z)+\frac{1+t}{\beta+\gamma}z\Phi'(z)$, then $L(z;0)=\phi(0)$.
If $L(z;t)=a_1(t)z+\cdots$ then

$$a_1(t)=\frac{\partial L(0;t)}{\partial z}=\left(1+\frac{t}{\beta+\gamma}\right)\Phi'(0)=1+\frac{t}{\beta+\gamma}$$

hence $\lim\limits_{t\to+\infty}|a_1(t)|=+\infty$ and since $Re\ (\beta+\gamma)>0$ we obtain $a_1(t)\neq 0$ for all $t\geq 0$.

In order to prove that $L(z;t)$ is a subordination chain we will use Lemma 2.2.
A simple computation shows that

$$(6)\qquad Re\ \left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right]=Re\ \left[\beta+\gamma+\frac{z\Phi''(z)}{\Phi'(z)}\right]+tRe\ \left[1+\frac{z\Phi''(z)}{\Phi'(z)}\right]$$

and we need to show that

$$(7)\qquad Re\ \left[1+\frac{z\Phi''(z)}{\Phi'(z)}\right]>0,\ z\in U$$

and

$$(8)\qquad Re\ \left[\beta+\gamma+\frac{z\Phi''(z)}{\Phi'(z)}\right]>0,\ z\in U.$$

Letting $q(z)=1+\frac{z\Phi''(z)}{\Phi'(z)}$ and by differentiating (5) we have

$$\phi'(z)=\left(1-\frac{1}{\beta+\gamma}\right)\Phi'(z)+\frac{1}{\beta+\gamma}\left(\Phi'(z)+z\Phi''(z)\right),$$

then by computing the logarithmical derivative of the above equality we deduce

$$(9)\qquad q(z)+\frac{zq'(z)}{q(z)+\beta+\gamma-1}=1+\frac{z\phi''(z)}{\phi'(z)}\equiv h(z).$$

Taking in Lemma 2.5 $\beta\equiv 1,\ \gamma\equiv\beta+\gamma-1$ since $h(0)=1=c$ then the condition $Re\ [\beta h(z)+\gamma]>0$ is equivalent to the assumption of the Theorem. It follows from Lemma 2.5 that the differential equation (9) has a solution $q\in H(U)$ with$q(0)=1$ and this solution verify $Re\ q(z)>1-\beta+\gamma,\ z\in U$. From $\beta+\gamma\leq 1$ we have $Re\, q(z)>1-\beta-\gamma\geq 0,\ z\in U$ hence inequality (7) is proved.

Using (7) and the fact that $\Phi \in A$, then Φ is univalent in U and from the above inequality we have

$$Re\left[\frac{z\Phi''(z)}{\Phi'(z)}+\beta+\gamma\right]=Re\ q(z)+\beta+\gamma-1>0,\ z\in U,$$

then relations (6) and (7) are proved and by using Lemma 2.2 we conclude that $L(z;t)$ is a subordination chain.

Now, by using Lemma 2.3 we will show that $\Psi(z) \prec \Phi(z)$. Without loss of generality we can assume that $\Phi(z)$ is regular and univalent in $\bar{U}$. If not, let $\psi_r(z)=\psi(rz)$, $\phi_r(z)=\phi(rz)$, $\Psi_r(z)=\Psi(rz)$ and $\Phi_r(z)=\Phi(rz)$, where $0<r<1$. Then Φ_r is regular and univalent in $\bar{U}$ and we need to prove that

$$\psi_r(z)\prec\phi_r(z) \text{ implies } \Psi_r(z)\prec\Phi_r(z), \text{ for all } 0<r<1$$

and by letting $r \longrightarrow 1^-$ we obtain $\Psi(z)\prec\Phi(z)$.

Suppose that $\Psi(z) \not\prec \Phi(z)$. Then by Lemma 2.3 there exist $z_0 \in U$, $t_0 \geq 0$ and$\zeta_0 \in \partial U$ such that $\Psi(z_0)=\Phi(\zeta_0)$, $z_0\Psi'(z_0)=(1+t_0)\zeta_0\Phi'(\zeta_0)$. We deduce that

$$L(\zeta_0;t_0)=\left(1-\frac{1}{\beta+\gamma}\right)\Phi(\zeta_0)+\frac{1+t_0}{\beta+\gamma}\zeta_0\Phi'(\zeta_0)=$$

$$=\left(1-\frac{1}{\beta+\gamma}\right)\Psi(z_0)+\frac{1}{\beta+\gamma}z_0\Psi'(z_0)=\psi(z_0),$$

and since $L(z;t)$ is a subordination chain and $\phi(z)=L(z;0)$ it follows that $\psi(z_0)=L(\zeta_0;t_0)\notin\phi(U)$ and this contradicts the assumption of the Theorem. □

Next we will presents a few particular cases of this Theorem obtained for appropriate choices of β, γ and $g(z)$.

COROLLARY 3.2. *Let* $f\in\mathcal{F}_{1,\gamma}$ *and* $g\in K(-\gamma)$ *where* $-1<\gamma\leq 0$. *Then*

$$f(z)\prec g(z) \textit{ implies } A_{1,\gamma}(f)(z)\prec A_{1,\gamma}(g)(z).$$

Proof. In order to use our Theorem for $\gamma=0$ we need to prove that

$$g\in K(-\gamma) \text{ implies } Re\ \frac{zg'(z)}{g(z)}>-\gamma,\ z\in U.$$

Letting $p(z) = \frac{1}{1+\gamma}\left[\frac{zg'(z)}{g(z)} + \gamma\right]$, since g is univalent then $g(z)/z \neq 0$, $z \in U$, hence $p \in H(U)$ and $p(0) = 1$. Twice differentiating the previous equality and using the fact that $g \in K(-\gamma)$ we have

$$(10) \qquad Re\left[p(z) + \frac{zp'(z)}{(1+\gamma)p(z) - \gamma}\right] > 0,\ z \in U.$$

Denoting by $\psi(w_1, w_2) = \frac{w_1 + w_2}{(1+\gamma)w_1 - \gamma}$ then

$$Re\ \psi(is,t) = Re\ \frac{-\gamma t}{\gamma^2 + (1+\gamma)^2 s^2} \leq 0 \text{ for all } s \in R \text{ and } t \leq -\frac{1}{2}(1+s^2).$$

From Lemma 2.4 we conclude that (10) implies $Re\ p(z) > 0$, $z \in U$, i.e.

$$Re\ \frac{zg'(z)}{g(z)} > -\gamma,\ z \in U \quad \text{so} \quad \frac{zg'(z)}{g(z)} + \gamma \prec R_{1+\gamma}, \quad \text{or} \quad g \in \mathcal{F}_{1,\gamma}.$$

□

For the case $\gamma = 0$, this result was obtained in [3] and later improved in [7] by the condition $g \in K(-1/2)$.

Taking $\beta + \gamma = 1$ in our Theorem we have :

COROLLARY 3.3. *Let* $\beta \in \mathbf{C}^*$, *let* $f, g \in \mathcal{F}_{\beta,1-\beta}$, *and for* $\beta \neq 1$ *suppose in addition that* $f(z)/z \neq 0$, $g(z)/z \neq O$, $z \in U$. *If* $\phi(z) = z\left[\frac{g(z)}{z}\right]^\beta \in K$, *then*

$$z\left[\frac{f(z)}{z}\right]^\beta \prec z\left[\frac{g(z)}{z}\right]^\beta \quad \textit{implies} \quad z\left[\frac{F(z)}{z}\right]^\beta \prec z\left[\frac{G(z)}{z}\right]^\beta$$

$$\textit{where } F = A_{\beta,1-\beta}(f),\ G = A_{\beta,1-\beta}(g).$$

For the case $g(z) = ze^{\lambda z}$ we may easily prove the next Corollary :

COROLLARY 3.4. *Let* $\beta, \gamma \in \mathbf{C}$ *with* $\beta \neq 0$, $0 < \beta + \gamma \leq 1$, *and let*

$$\lambda \in C \textit{ with } |\lambda| \leq \frac{2 + \beta + \gamma - \sqrt{(\beta+\gamma)^2 + 4}}{2|\beta|}.$$

Let $f \in \mathcal{F}_{\beta,\gamma}$ *and for* $\beta \neq 1$ *suppose in addition that* $\frac{f(z)}{z} \neq 0,\ z \in U$. *Then*

$$z\left[\frac{f(z)}{z}\right]^{\beta} \prec ze^{\lambda\beta z} \quad \textit{implies} \quad z\left[\frac{F(z)}{z}\right]^{\beta} \prec \frac{\beta+\gamma}{z^{\beta+\gamma-1}}\int_0^z t^{\beta+\gamma-1}e^{\lambda\beta t}\,dt$$

$$\textit{where} \quad F = A_{\beta,\gamma}(f).$$

Proof. For $g(z) = ze^{\lambda z}, \lambda \in \mathbf{C}$, we have $g \in \mathcal{F}_{\beta,\gamma}$ if and only if$h(z) = \lambda\beta z + \beta + \gamma \prec R_{\beta+\gamma}(z)$. But

$$|\lambda\beta| \leq \beta + \gamma + 1 \tag{11}$$

is equivalent to $|h(z)-(\beta+\gamma)| < \beta+\gamma+1,\ z \in U$, and this last condition is sufficient for $g \in \mathcal{F}_{\beta,\gamma}$. A simple calculus shows that $\phi(z) = ze^{\lambda\beta z}$ and $1 + z\frac{\phi''(z)}{\phi'(z)} = 1 + \lambda\beta z + \frac{\lambda\beta z}{1+\lambda\beta z}$ and in order to use our Theorem we must to determine the largest $r = |\lambda\beta|$ such that

$$Re\ \Phi(\zeta) > 1 - (\beta+\gamma), |\zeta| < r \text{ where } \Phi(\zeta) = 1 + \zeta + \frac{\zeta}{1+\zeta}.$$

Since $r \leq 1$ then $|\lambda\beta| \leq 1$ which implies (11). If $\zeta = re^{i\theta}, \theta \in [0, 2\pi]$ then

$$Re\ \Phi(re^{i\theta}) = 2 + r\cos\theta - \frac{1 + r\cos\theta}{r^2 + 2r\cos\theta + 1}.$$

It is easy to show that $Re\ \Phi(re^{i\theta}) \geq 2 - r - \frac{1}{1-r} = t(r),\ \theta \in [0, 2\pi]$ and

$$t(r) \geq 1 - (\beta+\gamma) \text{ if and only if } r \leq \frac{2+\beta+\gamma-\sqrt{(\beta+\gamma)^2+4}}{2}$$

$$= r_* \in (0,1)$$

or $|\lambda\beta| \leq r_*$. □

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