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ON THE STRUCTURE OF $V(ZD_4)$

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ABSTRACT. The group $V(ZD_4)$ of units of augmentation 1 in the integral group ring ZD_4 is characterized as the generalized free product of D_4 and D_4 with the centers amalgamated.

1. Introduction

The group of units of the integral group ring $U(ZD_4)$ was studied pretty well [1, 4, 5, 6, 7, 10]. In particular, in [10] J. S. Shin showed that

$$
\begin{aligned}
U(ZD_4) \cong \{(e, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in Z \oplus M_2(Z) | e = \pm 1, \delta = 1,& \\
a \equiv d \equiv \pm 1 (\text{mod } 4), b \equiv c \equiv 0 \text{ or } 2 (\text{mod } 4)\}& \\
\cup \{(e, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in Z \oplus M_2(Z) | e = \pm 1, \delta = 1,& \\
b \equiv -c \equiv \pm 1 (\text{mod } 4), a \equiv d \equiv 0 \text{ or } 2 (\text{mod } 4)\}& \\
\cup \{(e, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in Z \oplus M_2(Z) | e = \pm 1, \delta = -1,& \\
a \equiv -d \equiv \pm 1 (\text{mod } 4), b \equiv c \equiv 0 \text{ or } 2 (\text{mod } 4)\}& \\
\cup \{(e, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in Z \oplus M_2(Z) | e = \pm 1, \delta = -1,& \\
b \equiv c \equiv \pm 1 (\text{mod } 4), a \equiv d \equiv 0 \text{ or } 2 (\text{mod } 4)\},&
\end{aligned}
$$

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where $\delta = ad - bc$. Based on this result, we show that $V(ZD_4)$, the group of units of augmentation 1 in ZD_4, is isomorphic to the free product of D_4 with the amalgamated subgroup C_2, the cyclic group of order 2.

2. Theorems

From J. S. Shin's result mentioned above, we can easily show that

$$\begin{aligned} V(ZD_4) \cong G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\in GL(2,Z) | \delta = 1, \\ &a \equiv d \equiv \pm 1 (\text{mod } 4), b \equiv c \equiv 0 \text{ or } 2 (\text{mod } 4)\} \\ \cup \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\in GL(2,Z) | \delta = -1, \\ &b \equiv c \equiv \pm 1 (\text{mod } 4), a \equiv d \equiv 0 \text{ or } 2 (\text{mod } 4)\} \\ \cup \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\in GL(2,Z) | \delta = 1, \\ &b \equiv -c \equiv \pm 1 (\text{mod } 4), a \equiv d \equiv 0 \text{ or } 2 (\text{mod } 4)\} \\ \cup \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\in GL(2,Z) | \delta = -1, \\ &a \equiv -d \equiv \pm 1 (\text{mod } 4), b \equiv c \equiv 0 \text{ or } 2 (\text{mod } 4)\}, \end{aligned}$$

where $\delta = ad - bc$. Note that

$$x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

generate a subgroup of G which is isomorphic to D_4. For the convenience, we denote $< x, y >$ by D_4. First we show that D_4 has a torsion free normal complement in $V(ZD_4)$. This fact was first proved in [6].

LEMMA 2.1. *Let* $N = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z) \mid a \equiv d \equiv 1 (\text{mod } 4)$, $b \equiv c \equiv 0$ *or* $2 \mod 4)\}$. *Then N is a torsion free normal complement of* $D_4 =< x, y >$ *in* G.

Proof. First we note that if $\alpha, \beta \in N$, then $\alpha^{-1}\beta \in N$. Now let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N$. Then $x^{-1}\alpha x = \alpha^{-1} \in N$ and $y^{-1}\alpha y = \alpha \in N$. Hence D_4 normalizes N. Now we show that $G = D_4N$. By simple computations, we can see that $G = N \cup xN \cup x^2N \cup x^3N \cup yN \cup xyN \cup x^2yN \cup x^3yN = D_4N$. For example, if $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $ad - bc = 1, b \equiv -c \equiv 1 \pmod 4, a \equiv d \equiv 0$ or $2 \pmod 4$, then

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \in xN.$$

Now we show that N is torsion free. It is well known that any torsion element in $SL(2, Z)$ must have order 1, 2, 3, 4 or 6 [3]. Any element in $SL(2, Z)$ of order 3 is a conjugate of $U = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ or $U^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. But $U \notin N$. Hence N has no element of order 3. Now we show that N has no element of order 2. Let $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N$ be an element of order 2. Then

$$\beta^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

and $ad - bc = 1, ab + db = 0, a^2 + bc = bc + d^2 = 1, ac + cd = 0$. So, $b(a + d) = 0, c(a + d) = 0, a^2 + bc = a^2 + ad - 1 = 1$. Therefore $a^2 + ad = 2, a(a + d) = 2, a \neq 0, a + d \neq 0$. Hence $b = 0, c = 0$ and we must have $a = 1, d = 1$. But it is a contradiction. Hence N has no elements of order 4 or 6. Now it follows that N is torsion free. □

Now let $\delta = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}$, the elements in [7]. Observe that if we let $t = \delta\alpha$, then $t = \begin{pmatrix} 2 & 5 \\ -1 & -2 \end{pmatrix}$, $t^4 = 1$ and $\delta t \delta = t^3$ and hence $\langle \delta, \alpha \rangle = \langle t, \alpha \rangle \cong D_4$.

THEOREM 2.2. $G = \langle t, \alpha, x, y \rangle$

Proof. By Lemma 2.1, it suffices to show that $N < \langle t, \alpha, x, y\rangle$. Let $N_1 = \{\begin{pmatrix} 4k_1+1 & 4k_2 \\ 4k_3 & 4k_4+1 \end{pmatrix} \mid k_i \in Z\}$. Note that $N = N_1 \cup x^3yN$. Hence if $N_1 < G$ then $N < G$. And also note that for all $g \in G, N_1^g \subset N_1$, that is, G normalizes N_1. If $4k_2 = 0$ or $4k_3 = 0$, then $4k_1 + 1 = 4k_4 + 1 = 1$. So the matrices are of the types

$$\begin{pmatrix} 1 & 4k \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4k' & 1 \end{pmatrix}, k, k' \in Z.$$

But they are in G. So we may consider only matrices in N_1 whose entries are non-zero. Suppose $N_1 \not< G$ then we can choose an element $\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $N_1 - G$ with $|a| + |c|$ minimal. Note that

$$\begin{pmatrix} 1 & 0 \\ 4k & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 4ka+c & 4kb+d \end{pmatrix} \notin G$$

If $2|a| < |c|$, then by the division algorithm the integer k can be chosen so that $|4ka + c| < |c|$ and $|4ka + c| < |a| + |c|$, contradicting to the choice of η. If $a < |c| < 2|a|$ and a, c have same sign, consider

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a-c & 2b-d \\ -3a+2c & -3b+2d \end{pmatrix} \notin G.$$

Clearly $|2a - c| + |-3a + 2c| < |a| + |c|$, contradicting to the choice of η. Similarly we can show that a and c have different signs. Now only two cases of $\frac{1}{2}|a| < |c| < |a|$ and $|c| < \frac{1}{2}|a|$ remain. But for these cases similar arguments work to obtain the contradictions because the matrices $\begin{pmatrix} 1 & 4k \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$, and $\begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix}$ are in G. □

In [9], we define the concept of cornered matrices. We include it for the completeness.

DEFINITION. $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in G$ is called (i, j)-cornered, $1 \leqslant i, j \leqslant 2$, if

$$|a_{ij}| = \max\{|a_{11}|, |a_{12}|, |a_{21}|, |a_{22}|\} \text{ and } |a_{3-i3-j}| = \min\{|a_{11}|, |a_{12}|, |a_{21}|, |a_{22}|\},$$

and the absolute values of the sum and difference of the row and column containing a_{ij} is greater than or equal to those of the row and column containing a_{3-i3-j}, respectively.

Now consider

$$\Gamma = \{t^{i_1}\alpha^{j_1}x^{k_1}y^{l_1}\cdots t^{i_n}\alpha^{j_n}x^{k_n}y^{l_n} | i_m, j_m, k_m, l_m \\ = 0,1, i_m + j_m \neq 0, k_m + l_m \neq 0, m = 1, \cdots n, n > 0\}.$$

In the case of $n = 1$, there are nine matrices:

$$tx = \begin{pmatrix} -5 & 2 \\ 2 & -1 \end{pmatrix}, \alpha x = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}, t\alpha x = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix},$$
$$ty = \begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}, \alpha y = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, t\alpha y = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix},$$
$$txy = \begin{pmatrix} -5 & -2 \\ 2 & 1 \end{pmatrix}, \alpha xy = \begin{pmatrix} -4 & -1 \\ 1 & 0 \end{pmatrix}, t\alpha xy = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}.$$

All of above matrices are of the following three types :

$$T_1 : (-1)^r \begin{pmatrix} a & -b \\ -b & d \end{pmatrix}, \ (1,1) - \text{cornered},$$
$$T_2 : (-1)^r \begin{pmatrix} a & b \\ -b & -d \end{pmatrix}, \ (1,1) - \text{cornered},$$
$$T_3 : (-1)^r \begin{pmatrix} -b & a \\ d & -b \end{pmatrix}, \ (1,2) - \text{cornered, where } a, b, c, d \geqslant 0.$$

By simple computation, we can check that the set of matrices of above types is closed under multiplication. Hence we obtain the following:

THEOREM 2.3. *Any matrix* $X = t^{i_1}\alpha^{j_1}x^{k_1}y^{l_1}\cdots t^{i_n}\alpha^{j_n}x^{k_n}y^{l_n}$ *in* Γ *is one of the above three types.*

Observe that for any h, k, l, i, and j, $(-I)^h x^k y^l t^i \alpha^j$ is not of those three types in Theorem 2.3. Also note that $(-I)^h x^k y^l t^i \alpha^j = I$ if and only if $h = k = l = i = j = 0$. Applying von Dyck's theorem and the normal form theorem for the generalized free product [8], we can get the following:

THEOREM 2.4. *The group G is the free product of $\langle t,\alpha\rangle(\simeq D_4)$ and $\langle x,y\rangle(\simeq D_4)$ with the amalgamated subgroup $\langle t^2\rangle \simeq \langle x^2\rangle$.*

In [9], we discuss the normal complements in a free product. Here we consider the normal complements of D_4 in G. Consider a map $\phi : G \longrightarrow D_4$ defined by

$$\phi((-I)^h t^{i_1}\alpha^{j_1}x^{k_1}y^{l_1}\cdots t^{i_n}\alpha^{j_n}x^{k_n}y^{l_n}) = (-I)^h x^{i_1}y^{j_1}x^{k_1}y^{l_1}\cdots x^{i_n}y^{j_n}x^{k_n}y^{l_n},$$

where $h, i_m, j_m, k_m, l_m \in \{0,1\}, m = 1,\cdots,n$. Then ϕ is a well-defined homomorphism. Set $N = ker\phi$. If $\alpha \in G$ and $\phi(\alpha) = \beta \in D_4$, then $\phi(\alpha\beta^{-1}) = 1$ so $G = ND_4$. Furthermore $N\cap < t,\alpha >= N\cap < x,y >= 1$. Hence by the Karras-Solitar subgroup theorem for the generalized free product and the torsion-free subgroup property of G [2], N is a free group. Now consider $\theta : G \longrightarrow D_4$ defined by

$$\theta((-I)^h t^{i_1}\alpha^{j_1}x^{k_1}y^{l_1}\cdots t^{i_n}\alpha^{j_n}x^{k_n}y^{l_n}) = (-I)^h x^{k_1}y^{l_1}\cdots x^{k_n}y^{l_n},$$

where $h, i_m, j_m, k_m, l_m \in \{0,1\}, m = 1,\cdots,n$. Then θ is also a well-defined homomorphism. If we set $M = ker\theta$, then $G = MD_4$. But $< t,\alpha >\leq M$. Hence M is not torsion free.

THEOREM 2.5. *In $V(ZD_4)$, D_4 has both torsion free normal complement and non-torsion free normal complement.*

The similar result for $V(ZS_3)$ was mentioned in [9]. In [7], Polcino Milies proved that in $V(ZD_4)$ there are five conjugacy classes of elements of order 2. We remark that we can get this result easily from our characterization.

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