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SUBMANIFOLDS OF AN ALMOST QUATERNIONIC KAEHLER PRODUCT MANIFOLD

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ABSTRACT. We define an almost quaternionic Kaehler product manifold and study its submanifolds. Moreover we construct the curvature tensor of the product manifold of two quaternionic space forms.

1. Introduction

In [5], K. Yano and M. Kon studied submanifolds of Kaehlerian product manifolds. The Kaehlerian product of two Kaehlerian manifolds is also a Kaehlerian manifold. But the natural product manifold of two quaternionic Kaehler manifolds does not become a quaternionic Kaehler manifold. In this note, we define an almost quaternionic Kaehler product manifold and give an example. We also prove some theorems of submanifolds of almost quaternionic Kaehler product manifolds, and construct the curvature tensor of the product manifold of two quaternionic space forms.

2. Almost quaternionic Kaehler product manifolds

To begin with we define an almost product manifold (for details, see [cf. 6]). Let N be an n-dimensional manifold with a tensor F of type (1,1) such that

$$F^2 = I,$$

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where I denotes the identity transformation. Then we say that N is an *almost product manifold* with *almost product structure* F. If an almost product manifold N admits a Riemannian metric h such that

$$h(F\tilde{X}, F\tilde{Y}) = h(\tilde{X}, \tilde{Y})$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on N, then N is called to be an *almost product Riemannian manifold.*

DEFINITION. Let N be a $4n$-dimensional almost product Riemannian manifold with an almost product structure F and a 3-dimensional vector bundle E consisting of tensors of type $(1,1)$ over M satisfying the following conditions:

(a) In any coordinate neighborhood U of N, there exists a local basis of almost Hermitian structures $\theta_1, \theta_2, \theta_3$ of E such that

$$\begin{aligned} &\theta_s^2 = -I(\text{the identity transformation})(s = 1, 2, 3), \\ &\theta_1 \circ \theta_2 = -\theta_2 \circ \theta_1 = \theta_3, \theta_2 \circ \theta_3 = -\theta_3 \circ \theta_2 = \theta_1, \\ &\theta_3 \circ \theta_1 = -\theta_1 \circ \theta_3 = \theta_2. \end{aligned} \tag{2.1}$$

(b) There exist local 1-forms c_1, c_2 and c_3 on U such that

(2.2)

$$\begin{aligned} \tilde{\nabla}_{\tilde{X}}\theta_1 &= \lambda\{c_3(\tilde{X})\theta_2 - c_2(\tilde{X})\theta_3 + c_3(F\tilde{X})\theta_2 \circ F - c_2(F\tilde{X})\theta_3 \circ F\} \\ \tilde{\nabla}_{\tilde{X}}\theta_2 &= \lambda\{-c_3(\tilde{X})\theta_1 + c_1(\tilde{X})\theta_3 - c_3(F\tilde{X})\theta_1 \circ F + c_1(F\tilde{X})\theta_3 \circ F\} \\ \tilde{\nabla}_{\tilde{X}}\theta_3 &= \lambda\{c_2(\tilde{X})\theta_1 - c_1(\tilde{X})\theta_2 + c_2(F\tilde{X})\theta_1 \circ F - c_1(F\tilde{X})\theta_2 \circ F\} \end{aligned}$$

for some non-zero constant λ and any vector field $\tilde{X}$ on N, where F denotes an almost product structure on N and $\tilde{\nabla}$ the Levi-Civita connection of N.

In the case of a Riemannian manifold N, the vector bundle E satisfying (a) is called *almost quaternionic structure* in N. Such a manifold N is called *almost quaternionic manifold.* If an almost product Riemannian manifold N with an almost product structure F satisfies the condition (a) and (b), then N is called *almost quaternionic Kaehler* *product manifold* and the bundle E is called an *almost quaternionic Kaehler product structure.*

Now we give an example of an almost quaternionic Kaehler product manifold.

Let ${N_1}^{4n_1}$ be a $4n_1$-dimensional quaternionic Kaehler manifold with metric h_1. Then there exists a 3-dimensional vector bundle E_1 of tensors of type $(1,1)$ such that in any coordinate neighborhood U_1 of ${N_1}^{4n_1}$, there exists a local basis of almost Hermitian structures ϕ_1, ϕ_2, ϕ_3 of E_1 satisfying

$$
\begin{aligned}
&\phi_s^2 = -I(\text{the identity transformation})(s = 1, 2, 3),\\
&\phi_1 \circ \phi_2 = -\phi_2 \circ \phi_1 = \phi_3, \phi_2 \circ \phi_3 = -\phi_3 \circ \phi_2 = \phi_1,\\
&\phi_3 \circ \phi_1 = -\phi_1 \circ \phi_3 = \phi_2,
\end{aligned}
\tag{2.3}
$$

and there exist local 1-forms a_1, a_2 and a_3 on U_1 satisfying

$$
\begin{aligned}
{}^1\nabla_X \phi_1 &= a_3(X)\phi_2 - a_2(X)\phi_3\\
{}^1\nabla_X \phi_2 &= -a_3(X)\phi_1 + a_1(X)\phi_3\\
{}^1\nabla_X \phi_3 &= a_2(X)\phi_1 - a_1(X)\phi_2
\end{aligned}
\tag{2.4}
$$

for any vector field X on ${N_1}^{4n_1}$, where ${}^1\nabla$ the Levi-Civita connection of ${N_1}^{4n_1}$. The bundle E_1 satisfying (2.3) and (2.4) is called a *quaternionic Kaehler structure* in N_1 (cf. [2, 3, 6]).

Let ${N_2}^{4n_2}$ be another quaternionic Kaehler manifold with metric h_2. Assume that a local basis of almost Hermitian structures ψ_1, ψ_2, ψ_3 of a 3-dimensional vector bundle E_2 of tensors of type $(1,1)$ satisfy the above algebraic relation (2.3), and there exist local 1-forms b_1, b_2 and b_3 in a coordinate neighborhood U_2 of ${N_2}^{4n_2}$ such that

$$
\begin{aligned}
{}^2\nabla_X \psi_1 &= b_3(X)\psi_2 - b_2(X)\psi_3\\
{}^2\nabla_X \psi_2 &= -b_3(X)\psi_1 + b_1(X)\psi_3\\
{}^2\nabla_X \psi_3 &= b_2(X)\psi_1 - b_1(X)\psi_2
\end{aligned}
$$

for any vector field X on ${N_2}^{4n_2}$, where ${}^2\nabla$ the Levi-Civita connection of ${N_2}^{4n_2}$.

Now we consider a product manifold $N := N_1{}^{4n_1} \times N_2{}^{4n_2}$ of two quaternionic Kaehler manifolds N_1 and N_2. We denote by P and Q the projection operators of tangent space of N to the tangent space of N_1 and N_2 respectively. Then we have

$$P^2 = P,\ Q^2 = Q,\ PQ = 0 = QP.$$

Setting $F = P - Q$, then we obtain $F^2 = I$, i.e., F is an almost product structure on N. Moreover, we define a Riemannian metric h on N by

$$h(\tilde{X}, \tilde{Y}) = h_1(P\tilde{X}, P\tilde{Y}) + h_2(Q\tilde{X}, Q\tilde{Y})$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ of N. It also follows that

$$h(F\tilde{X}, \tilde{Y}) = h(F\tilde{Y}, \tilde{X}).$$

For any vector field $\tilde{X}$ on N we put

$$\theta_s\tilde{X} = \phi_s P\tilde{X} + \psi_s Q\tilde{X},\ s = 1, 2, 3. \tag{2.5}$$

Now we consider the vector bundle E over N generated by $\{\theta_s = \phi_s \oplus \psi_s : s = 1, 2, 3\}$, where $\{\phi_s : s = 1, 2, 3\}$ and $\{\psi_s : s = 1, 2, 3\}$ are local bases of quaternionic Kaehler structures E_1 and E_2 respectively. Then, for any local coordinate neighborhood $U_1 \times U_2$, we see that the local basis of almost Hermitian structures $\theta_1, \theta_2, \theta_3$ satisfies the algebraic relation (a). We also define local 1-forms c_1, c_2 and c_3 on $U_1 \times U_2$ by

$$c_s(\tilde{X}) = a_s(P\tilde{X}) + b_s(Q\tilde{X}),\ s = 1, 2, 3 \tag{2.6}$$

for any vector field $\tilde{X}$ on N. Then, for the induced Levi-Civita connection $\tilde{\nabla}$ on N defined by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = {}^1\nabla_{P\tilde{X}}P\tilde{Y} + {}^2\nabla_{Q\tilde{X}}Q\tilde{Y},$$

the covariant differentiation (b) with $\lambda = \frac{1}{2}$ holds. Moreover we know that

$$P\theta_s = \phi_s P,\ Q\theta_s = \psi_s Q. \tag{2.7}$$

Summing up, we obtain

PROPOSITION 2.1. *The product manifold $N := N_1{}^{4n_1} \times N_2{}^{4n_2}$ of two quaternionic Kaehler manifolds N_1 and N_2 is an almost quaternion Kaehler product manifold.*

3. F-invariant and F-anti-invariant submanifolds

Let M be an m-dimensional manifold isometrically immersed in a $4n$- dimensional almost quaternionic Kaehler product manifold N. M is called *F-invariant* (resp. *F-anti-invariant*) if $FT_xM \subset T_xM$ (resp. $FT_xM \subset T_xM^\perp$, where $T_xM^\perp$ denotes the normal space of T_xM in T_xN) for each point $x \in M$. It is known [cf. 6] that if M is F-invariant in the product manifold $N := {N_1}^{4n_1} \times {N_2}^{4n_2}$ of two quaternionic Kaehler manifolds N_1 and N_2, then M is a Riemannian product manifold $M = M_1 \times M_2$, where M_1 is a submanifold of ${N_1}^{4n_1}$ and M_2 is a submanifold of ${N_2}^{4n_2}$, and M_1 and M_2 being both totally geodesic in M.

A submanifold M in an almost quaternionic manifold N with an almost quaternionic structure E is called (i) *invariant* if $\theta T_xM \subset T_xM$ for any $\theta \in E$, (ii) *anti-invariant* (or *totally real*) if $\theta T_xM \subset T_xM^\perp$ for any $\theta \in E$ and (iii) *totally complex* if there exists a one-dimensional subbundle E° of E such that $\theta T_xM \subset T_xM$ for $\theta \in E^\circ$ and $\theta T_xM \subset T_xM^\perp$ for $\theta \perp E^\circ$ for each $x \in M$ (cf. [1, 4]).

THEOREM 3.1. *Let M be an F-invariant, invariant submanifold of an almost quaternionic Kaehler product manifold $N = N_1 \times N_2$. Then M is a Riemaannian product manifold $M = M_1 \times M_2$, where M_1 and M_2 are invariant submanifolds of N_1 and N_2, respectively.*

Proof. Assume that M is an invariant submanifold of N. Since M is F-invariant, M is a Riemannian product manifold $M_1 \times M_2$, where M_1 is a submanifold of N_1 and M_2 is a submanifold of N_2. We now show that M_1 and M_2 are invariant in N_1 and N_2, respectively. Let $\{\theta_s = \phi_s \oplus \psi_s; s = 1, 2, 3\}$ be a local basis of almost Hermitian structurs of E as in (2.6). Let $X \in T_xM_1$. Then for $s = 1, 2, 3$

$$\theta_s X = \phi_s PX + \psi_s QX = \phi_s X \in T_xN_1 \cap T_xM = T_xM_1.$$

Therefore M_1 is invariant in N_1. Similarly, M_2 is invariant in N_2. □

THEOREM 3.2. *Let M be an F-invariant, totally real submanifold of an almost quaternionic Kaehler product manifold $N = N_1 \times N_2$. Then M is a Riemannian product manifold $M_1 \times M_2$, where M_1 and M_2 are totally real submanifolds of N_1 and N_2, respectively.*

Proof. M is a Riemannian product manifold $M_1 \times M_2$ because of F-invariance. Let $X \in T_xM_1$. Then we have for $s = 1, 2, 3$

$$\theta_s X = \phi_s PX + \psi_s QX = \phi_s PX \in T_xM^{\perp}.$$

It is clear from (2.7) that $Q\theta_s X = \psi_s QX = 0$. Hence we see that $Q\phi_s PX = 0$. This means that $\phi_s PX \in T_xN_1$. Thus M_1 is totally real in N_1. In the same way we know that M_2 is totally real in N_2. □

THEOREM 3.3. *Let M be an F-invariant, totally complex submanifold of an almost quaternionic Kaehler product manifold $N = N_1 \times N_2$. Then M is a Riemannian product manifold $M_1 \times M_2$, where M_1 and M_2 are totally complex submanifolds of N_1 and N_2, respectively.*

Proof. Since M is totally complex in N, there exists a one-dimensional subbundle E° of E such that $\theta T_xM \subset T_xM$ for each $x \in M$. Then θ is of the form $\theta = \lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\theta_3$, where λ_1, λ_2 and λ_3 are some smooth functions on N. If $X \in T_xM_1$, then

$$\theta X = \lambda_1(x)\phi_1 X + \lambda_2(x)\phi_2 X + \lambda_3(x)\phi_3 X \in T_xM_1.$$

Put $E_1^\circ := \text{span}\{\theta|_{M_1}\}$. Then E_1° forms a one-dimensional subbundle of E_1. Next take any $\eta \in E$ such that $\eta \perp E^\circ$ and $\eta T_xM \subset T_xM^{\perp}$ for each $x \in M$. Put $\eta = \sum_s^3 \mu_s\theta_s$ for some smooth functions $\mu_s, s = 1, 2, 3$ on N. If $X \in T_xM$, then $\eta X = \sum_s \mu_s(x)\psi_s X \in T_xM^{\perp}$. On the other hand $Q\eta X = \sum_s \mu_s Q\psi_s X = \sum_s \mu_s\psi_s QX = 0$. Thus $\eta X \in T_xN_1 \cap T_xM^{\perp} = T_xM_1^{\perp}$ for each $x \in M$. This means that M_1 is totally complex in N_1. Similarly M_2 is also totally complex in N_2. We complete the proof. □

Let ${N_1}^{4n_1}$ be a $4n_1$-dimensional quaternionic Kaehler manifold with a local basis $\{\phi_1, \phi_2, \phi_3\}$ of E_1 Let $Q(X)$ be the so-called *quaternionic section* determined by X, which is a 4-plane spanned by $\{X, \phi_s X : s = 1, 2, 3\}$, where X is a unit vector on N_1. Any 2-plane in a quaternionic section is called a *quaternionic plane.* The sectional curvature of a quaternionic plane π is called the *quaternionic sectional curvature* of π. A quaternionic Kaehler manifold is a *quaternionic space form* if its quaternionic sectional curvatures are equal to a constant.

It is well known that a quaternionic Kaehler manifold N_1 is a quaternionic space form with constant quaternionic sectional curvature λ_1 if and only if its curvature tensor R_1 is of the following form (cf. [3], [6]) :

$$R_1(X,Y)Z = \frac{\lambda_1}{4}\big[h_1(Y,Z)X - h_1(X,Z)Y + \sum_s \{h_1(\phi_s Y, Z)\phi_s X - h_1(\phi_s X, Z)\phi_s Y - 2h_1(\phi_s X, Y)\phi_s Z\}\big]$$

where X, Y and Z are vector fields on N_1.

Here and in the sequel, we denote by ${N_1}^{4n_1}(\lambda_1)$ the $4n_1$-dimensional quaternionic space form of constant quaternionic sectional curvature λ_1.

Let ${N_2}^{4n_2}(\lambda_2)$ be a $4n_2$-dimensional quaternionic space form with constant quaternionic sectional curvature λ_2 and a local basis $\{\psi_1, \psi_2, \psi_3\}$ of E_2. Then the curvature tensor R_2 of N_2 is given by the

$$R_2(X,Y)Z = \frac{\lambda_2}{4}\big[h_2(Y,Z)X - h_2(X,Z)Y + \sum_s \{h_2(\psi_s Y, Z)\psi_s X - h_2(\psi_s X, Z)\psi_s Y - 2h_2(\psi_s X, Y)\psi_s Z\}\big]$$

where X, Y and Z are vector fields on N_2.

Now we consider an almost quaternionic Kaehler product manifold $N = {N_1}^{4n_1}(\lambda_1) \times {N_2}^{4n_2}(\lambda_2)$ of quaternionic space forms ${N_1}^{4n_1}(\lambda_1)$ and ${N_2}^{4n_2}(\lambda_2)$. Then the curvature tensor R_h of $N = {N_1}^{4n_1}(\lambda_1) \times {N_2}^{4n_2}(\lambda_2)$ is given by

(3.1)
$$\begin{aligned} R_h(\tilde X,\tilde Y)\tilde Z = \alpha\big[&h(\tilde Y,\tilde Z)\tilde X - h(\tilde X,\tilde Z)\tilde Y + h(F\tilde Y,\tilde Z)F\tilde X - h(F\tilde X,\tilde Z)F\tilde Y \\ &+ \sum_s \{h(\theta_s\tilde Y,\tilde Z)\theta_s\tilde X - h(\theta_s\tilde X,\tilde Z)\theta_s\tilde Y - 2h(\theta_s\tilde X,\tilde Y)\theta_s\tilde Z\} \\ &+ \sum_s \{h(F\theta_s\tilde Y,\tilde Z)F\theta_s\tilde X - h(F\theta_s\tilde X,\tilde Z)F\theta_s\tilde Y - 2h(F\theta_s\tilde X,\tilde Y)F\theta_s\tilde Z\}\big] \end{aligned}$$

$$
\begin{aligned}
&+\beta[h(F\tilde{Y},\tilde{Z})\tilde{X}-h(F\tilde{X},\tilde{Z})\tilde{Y}+h(\tilde{Y},\tilde{Z})F\tilde{X}-h(\tilde{X},\tilde{Z})F\tilde{Y}\\
&+\sum_s\{h(F\theta_s\tilde{Y},\tilde{Z})\theta_s\tilde{X}-h(F\theta_s\tilde{X},\tilde{Z})\theta_s\tilde{Y}+h(\theta_s\tilde{Y},\tilde{Z})F\theta_s\tilde{X}\\
&-h(\theta_s\tilde{X},\tilde{Z})F\theta_s\tilde{Y}-2h(F\theta_s\tilde{X},\tilde{Y})\theta_s\tilde{Z}-2h(\theta_s\tilde{X},\tilde{Y})F\theta_s\tilde{Z}\}]
\end{aligned}
$$

for any vector fields $\tilde{X},\tilde{Y}$ and $\tilde{Z}$ on N, where $F := P-Q$ is an almost product structure on N as in section 2, $\alpha := \dfrac{\lambda_1+\lambda_2}{16}$ and $\beta := \dfrac{\lambda_1-\lambda_2}{16}$.

REMARK 3.4. In the product manifold $N = {N_1}^{4n_1}\ (\lambda_1)\times {N_2}^{4n_2}$ (λ_2), if $n_1=n_2$ and $\lambda_1=\lambda_2$, then N is an Einstein manifold. In fact, The Ricci tensor ρ_h of N is given by

$$
\begin{aligned}
\rho_h(\tilde{X},\tilde{Y}) = &\ \alpha\{(4(n+4)h(\tilde{X},\tilde{Y})+(Tr_hF)h(F\tilde{X},\tilde{Y})\}\\
&+\beta\{(4(n+4)h(F\tilde{X},\tilde{Y})+(Tr_hF)h(\tilde{X},\tilde{Y})\}
\end{aligned}
$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on N. If $\lambda_1=\lambda_2$ (i.e., $\beta=0$) and $n_1=n_2$ (i.e., $Tr_hF=0$), then $\rho_h = 4(n+4)\alpha h$. Hence N is an Einstein manifold.

For a submanifold M in an almost quaternionic Kaehler product manifold N we denote by h the metric tensor of M as well as that of N. Let ∇ be the induced Levi-Civita connection on M. The Gauss and Weingarten formulas for M are respectively given by

$$\tilde{\nabla}_XY=\nabla_XY+B(X,Y), \tag{3.2}$$

$$\tilde{\nabla}_X\xi=-A_\xi X+D_X\xi \tag{3.3}$$

for any vector fields X,Y tangent to M and any vector field ξ normal to M, where B, A_ξ and D are the second fundamental form, the second fundamental tensor associated with ξ and the normal connection, respectively. Moreover, B and A_ξ are related with $h(A_\xi X,Y) = h(B(X,Y),\xi)$.

For the second fundamental form B, we define the covariant differentiation $\bar{\nabla}$ with respect to the connection in $TM\oplus TM^\perp$ by

$$(\bar{\nabla}_XB)(Y,Z)=D_XB(Y,Z)-B(\nabla_XY,Z)-B(Y,\nabla_XZ)$$

for X, Y and Z tangent to M. Then the Gauss, Codazzi and Ricci equations of M are given by

(3.4)
$$h(K(X,Y)Z,W) = h(R_h(X,Y)Z,W) + h(B(Y,Z),B(X,W)) - h(B(X,Z),B(Y,W)),$$

$$(3.5) \qquad (R_h(X,Y)Z)^{\perp} = (\bar{\nabla}_X B)(Y,Z) - (\bar{\nabla}_Y B)(X,Z),$$

$$(3.6) \qquad h(R_h(X,Y)\xi,\eta) = h(K^{\perp}(X,Y)\xi,\eta) - h([A_\xi, A_\eta]X,Y)$$

for X, Y, Z, W tangent to M and ξ, η normal to M, where K and $K^{\perp}$ are the curvature tensors associated with ∇ and D respectively, and $\perp$ in (3.5) denotes the normal component.

Let M be an m-dimensional F-invariant, totally real submanifold of an almost quaternionic Kaehler product manifold $N = {N_1}^{4n_1}(\lambda_1) \times {N_2}^{4n_2}(\lambda_2)$. Then the equations (3.4) and (3.5) with (3.1) are respectively transformed into the following forms.

(3.7)
$$\begin{aligned} h(K(X,Y),W) = {} & \alpha\{h(Y,Z)h(X,W) - h(X,Z)h(Y,W) \\ & + h(FY,Z)h(FX,W) - h(FX,Z)h(FY,W)\} \\ & + \beta\{h(FY,Z)h(X,W) - h(FX,Z)h(Y,W) \\ & + h(Y,Z)h(FX,W) - h(X,Z)h(FY,W)\} \\ & + h(B(Y,Z),B(X,W)) - h(B(X,Z),B(Y,W)), \end{aligned}$$

$$(3.8) \qquad (\bar{\nabla}_X B)(Y,Z) = (\bar{\nabla}_Y B)(X,Z)$$

From (3.7) we see that the Ricci tensor ρ_M of M is given by

(3.9)
$$\begin{aligned} \rho_M(X,Y) = {} & \alpha\{(m-2)h(X,Y) + h(FX,Y)(TrF)\} \\ & + \beta\{(m-2)h(FX,Y) + h(X,Y)(TrF)\} \\ & + \sum_{i=1}^{m}\{h(B(X,Y),B(e_i,e_i)) - h(B(e_i,X),B(e_i,Y))\}, \end{aligned}$$

where $\{e_i; i = 1, ..., m\}$ is an orthonormal frame of M and $TrF := \sum_{i=1}^{m} h(Fe_i, e_i)$. Therefore the scalar curvature τ_M of M is given by

(3.10)
$$\begin{aligned}\tau_M &= \alpha\{m(m-2) + (TrF)^2\} + 2(m-1)\beta(TrF)\\ &+ \sum_{i,j}\{h(B(e_j, e_j), B(e_i, e_i)) - h(B(e_i, e_j), B(e_i, e_j))\}.\end{aligned}$$

From (3.7) we have

PROPOSITION 3.5. *Let M be an m-dimensional F-invariant, totally real submanifold of an almost quaternionic Kaehler product manifold $N = {N_1}^{4n_1}(\lambda_1) \times {N_2}^{4n_2}(\lambda_2)$. If M is totally geodesic, then $M = M_1^{m_1}(\frac{\lambda_1}{4}) \times M_2^{m_2}(\frac{\lambda_2}{4})$, where $M_1^{m_1}(\frac{\lambda_1}{4})$ and $M_2^{m_2}(\frac{\lambda_2}{4})$ $(m_1 + m_2 = m)$ are real space forms of constant curvatures $\frac{\lambda_1}{4}$ and $\frac{\lambda_2}{4}$, respectively.*

PROPOSITION 3.6. *Let $M = M_1^p \times M_2^p$ be a $2p$- dimensional F-invariant, totally real minimal submanifold of $N = N_1^{4p}(\lambda) \times N_2^{4p}(\lambda)$. Then M is totally geodesic if and only if M satisfies one of the following conditions:*
(a) M is a Riemannian product manifold $M_1^p(\frac{\lambda}{4}) \times M_2^p(\frac{\lambda}{4})$,
(b) $\rho_M = \frac{\lambda}{4}(n-1)h$,
(c) $\tau_M = \frac{\lambda}{2}n(n-1)$.

Proof. It is clear from (3.7), (3.9) and (3.10). □

LEMMA [1]. *Let W^{4n} be a quaternionic Hermitian vector space with positive definite inner product $<\ ,\ >$ and quaternionic structure $\{\theta_1, \theta_2, \theta_3\}$. Let $W^m (m \geq 4)$ be an m-dimensional linear subspace of W^{4n}. Then W^m satisfies the property*

$$\sum_{s}^{3} < X, \theta_s Y > \theta_s Y \in W^m$$

for any vectors X and Y in W^m if and only if W^m is one of the following:
(1) W^m is an invariant subspace of W^{4n},
(2) W^m is a totally real subspace of W^{4n},

(3) *W^m is a totally complex subspace of W^{4n}.*

If, for any vectors X, Y and Z tangent to M, $R_h(X,Y)Z$ is also tangent to M, i.e., $R_h(X,Y)T_xM \subset T_xM$ for each $x \in M$, then M is said to be *curvature invariant.*

THEOREM 3.7. *Let M be an (m_1+m_2)-dimensional F-invariant submanifold of an almost quaternionic Kaehler product manifold $N = N_1^{n_1}(\lambda) \times N_2^{n_2}(\lambda)$ ($\lambda \neq 0$, and $m_1, m_2 \geq 4$). If M is curvature invariant, then M is a Riemannian product manifold $M_1^{m_1} \times M_2^{m_2}$ such that*
(i) $M_1^{m_1}$ is invariant or totally complex or totally real in $N_1^{n_1}(\lambda)$,
(ii) $M_2^{m_2}$ is invariant or totally complex or totally real in $N_2^{n_2}(\lambda)$.

Proof. Since $\lambda \neq 0$, (3.1) with F-invariance gives

$$\sum_s \{h(\theta_s Y, Z)\theta_s X - h(\theta_s X, Z)\theta_s Y - 2h(\theta_s X, Y)\theta_s Z$$
$$+h(F\theta_s Y, Z)F\theta_s X - h(F\theta_s X, Z)F\theta_s Y - 2h(F\theta_s X, Y)F\theta_s Z\} \in T_xM$$

for any vector fields X, Y and Z tangent to M. Putting $Y = Z$ in this expression, we obtain

$$\sum_s \{h(X, \theta_s Y)\theta_s Y + h(X, \theta_s FY)\theta_s FY\} \in T_xM.$$

Since M is F-invariant, M is of the form $M_1^{m_1} \times M_2^{m_2}$. If $X \in T_xM_1^{m_1}$, then $FX = X$ and if $X \in T_xM_2^{m_2}$, then $FX = -X$. Now let $X, Y \in T_xM_1^{m_1}$. Then for each $x \in M$

$$\sum_s h(X, \theta_s Y)\theta_s Y = \sum_s h(X, \phi_s Y)\phi_s Y \in T_xM_1^{m_1}.$$

Hence Lemma implies that $M_1^{m_1}$ is invariant or totally complex or totally real submanifold of $N_1^{n_1}(\lambda)$. In the same way, for $X, Y \in T_xM_2^{m_2}$ and each $x \in M$ we obtain

$$\sum_s h(X, \theta_s Y)\theta_s Y = \sum_s h(X, \psi_s Y)\psi_s Y \in T_xM_2^{m_2}.$$

Again combining this with Lemma, we complete the proof. □

THEOREM 3.8. *Let M be an invariant submanifold of an almost quaternionic Kaehler product manifold $N = N_1^{n_1}(\lambda) \times N_2^{n_2}(\lambda)$ $(\lambda \neq 0)$. If M is curvature invariant, then M is F-invariant or F-anti-invariant.*

Proof. Assume that M is an invariant and curvature invariant submanifold of N. Then, for any vector fields X, Y and Z tangent to M, (3.1) implies

(3.11)
$$\sum_s \{h(F\theta_s Y, Z)F\theta_s X - h(F\theta_s X, Z)F\theta_s Y - 2h(F\theta_s X, Y)F\theta_s Z\} + h(FY, Z)FX - h(FX, Z)FY \in T_xM.$$

Putting $Y = Z$ in (3.11), we find

$$h(FY, Y)FX - h(FX, Y)FY + 3\sum_s h(FX, \theta_s Y)F\theta_s Y \in T_xM. \tag{3.12}$$

Replacing Y by $\theta_1 Y, \theta_2 Y$ and $\theta_3 Y$ in turns in (3.12), we obtain (3.13) $\sim$ (3.15) respectively

$$\begin{aligned} 3\{h(FX, Y)FY + h(FX, \theta_3 Y)F\theta_3 Y + h(FX, \theta_2 Y)F\theta_2 Y\} \\ + h(FY, Y)FX - h(FX, \theta_1 Y)F\theta_1 Y \in T_xM, \end{aligned} \tag{3.13}$$

$$\begin{aligned} 3\{h(FX, Y)FY + h(FX, \theta_3 Y)F\theta_3 Y + h(FX, \theta_1 Y)F\theta_1 Y\} \\ + h(FY, Y)FX - h(FX, \theta_2 Y)F\theta_2 Y \in T_xM, \end{aligned} \tag{3.14}$$

$$\begin{aligned} 3\{h(FX, Y)FY + h(FX, \theta_2 Y)F\theta_2 Y + h(FX, \theta_1 Y)F\theta_1 Y\} \\ + h(FY, Y)FX - h(FX, \theta_3 Y)F\theta_3 Y \in T_xM. \end{aligned} \tag{3.15}$$

Then (3.13) + (3.14) + (3.15) yields
(3.16)
$$3h(FY, Y)FX + 9h(FX, Y)FY + 5\sum_s h(FX, \theta_s Y)F\theta_s Y \in T_xM.$$

Next, (3.12) × 5 − (3.16) × 3 gives

$$h(FY,Y)FX - 8h(FX,Y)FY \in T_xM. \tag{3.17}$$

Putting $X = Y$ in (3.17), we get

$$h(FY,Y)FY \in T_xM,$$

which implies that $FY \in T_xM$ or $h(FY,Y) = 0$. If $h(FY,Y) = 0$, then (3.17) implies that $h(FX,Y)FY \in T_xM$ and hence $FY \in T_xM$ or $FY \in T_xM^\perp$. Consequently we see that for any $Y \in T_xM$, $FY \in T_xM$ or $FY \in T_xM^\perp$. Thus we have $FT_xM \subset T_xM$ or $FT_xM \subset T_xM^\perp$ for each point $x \in M$. Therefore we complete the proof. □

COROLLARY 3.9. *Let M be an invariant submanifold of an almost quaternionic Kaehler product manifold $N = N_1^{n_1}(\lambda) \times N_2^{n_2}(\lambda)$ $(\lambda \neq 0)$. If M is totally geodesic, then M is F-invariant or F-anti-invariant.*

Proof. It follows from the fact that if M is totally geodesic, then M is curvature invariant. □

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