$k$-NIL RADICAL IN BCI-ALGEBRAS II

Y. B. JUN, S. M. HONG AND E. H. ROH

ABSTRACT. This paper is a continuation of [3]. We prove that if $A$ is a quasi-associative (resp. an implicative) ideal of a BCI-algebra $X$ then the $k$-nil radical of $A$ is a quasi-associative (resp. an implicative) ideal of $X$. We also construct the quotient algebra $X/\langle A; k \rangle$ of a BCI-algebra $X$ by the $k$-nil radical $\langle A; k \rangle$, and show that if $A$ and $B$ are closed ideals of BCI-algebras $X$ and $Y$ respectively, then

$$X/\langle A; k \rangle \times Y/\langle B; k \rangle \cong X \times Y/\langle A \times B; k \rangle.$$  

By a BCI-algebra we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the axioms:

(I) $(x * y) * (x * z) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = y * x = 0$ implies $x = y$,

for all $x, y$ and $z$ in $X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. A BCI-algebra $X$ is said to be $p$-semisimple if $\{x \in X | 0 \leq x\} = \{0\}$.

In any BCI-algebra $X$, the following hold:

1. $x * 0 = x$.

2. $(x * y) * z = (x * z) * y$.

3. $0 * (0 * (0 * x)) = 0 * x$.

4. $0 * (x * y) = (0 * x) * (0 * y)$.

Received October 6, 1996. Revised April 10, 1997.

1991 Mathematics Subject Classification: 03G25, 06F35.

Key words and phrases: $k$-nil radical, (closed) ideal, quasi-associative ideal, implicative ideal, quotient algebra.

The first author was supported by the LG Yonam Foundation (1995), and the present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1995, Project No. BSRI-95-1406.
In what follows, $X$ would mean a BCI-algebra unless otherwise specified.

A non-empty subset $A$ of $X$ is called a subalgebra of $X$ if $x \ast y \in A$ whenever $x, y \in A$.

A non-empty subset $A$ of $X$ is called an ideal of $X$ if $0 \in A$ and if $x \ast y, y \in A$ imply that $x \in A$. We note that if $x$ is in an ideal $A$ of $X$ and $y \leq x$, then $y \in A$.

An ideal $I$ of $X$ is said to be closed if $0 \ast x \in I$ whenever $x \in I$. We note that every closed ideal of $X$ is a subalgebra of $X$.

A mapping $f : X \rightarrow Y$ of BCI-algebras is called a homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$.

For any elements $x, y$ in $X$, let us write $x \ast y^k$ for $(((x \ast y) \ast y) \ast y) \ast y$ where $y$ occurs $k$ times.

**Lemma 1** (Huang [2, Lemmas 1 and 2]). For any $x, y$ in $X$ and any positive integer $k$, we have

(i) $0 \ast (x \ast y)^k = (0 \ast x^k) \ast (0 \ast y^k)$.

(ii) $0 \ast (0 \ast x)^k = 0 \ast (0 \ast x^k)$.

**Definition 1** (Hong et al. [3]). Let $A$ be a subset of $X$. For given positive integer $k$, the $k$-nil radical of $A$, denote by $[A; k]$, is the set of all elements of $X$ satisfying $0 \ast x^k \in A$, i.e.,

$$[A; k] := \{x \in X : 0 \ast x^k \in A\}.$$ 

Note that $[A; k]$ may not contain $A$ itself (see [3]).

**Lemma 2** (Hong et al. [3, Proposition 2]). Let $A$ be a subalgebra of $X$ and $k$ a positive integer. Then

(i) if $x \in [A; k]$, then $0 \ast x \in [A; k]$.

(ii) if $x \ast y \in [A; k]$, then $y \ast x \in [A; k]$.

**Lemma 3** (Hong et al. [3, Theorem 1]). If $A$ is a subalgebra of $X$, then the $k$-nil radical of $A$ is a subalgebra of $X$ containing $A$ for every positive integer $k$.

**Lemma 4** (Hong et al. [3, Theorems 2 and 3]). If $A$ is a (closed) ideal of $X$, then the $k$-nil radical of $A$ is a (closed) ideal of $X$ for every positive integer $k$. 
Definition 2 (Yue et al. [8]). A non-empty subset $A$ of $X$ is called a quasi-associative ideal of $X$ if it satisfies

(i) $0 \in A$,

(ii) $x \ast (y \ast z) \in A$ and $y \in A$ imply $x \ast z \in A$,

for all $x, y, z \in X$.

Theorem 1. If $A$ is a quasi-associative ideal of $X$, then so is the $k$-nil radical of $A$ for every positive integer $k$.

Proof. Clearly $0 \in [A; k]$. Let $x, y, z \in X$ be such that $x \ast (y \ast z) \in [A; k]$ and $y \in [A; k]$. By using Lemma 1(i), we obtain

$$0 \ast (x \ast (y \ast z))^k = (0 \ast x^k) \ast (0 \ast (y \ast z)^k)$$

$$= (0 \ast x^k) \ast ((0 \ast y^k) \ast (0 \ast z^k)) \in A$$

and $0 \ast y^k \in A$. Since $A$ is a quasi-associative ideal, it follows from Lemma 1(i) that

$$0 \ast (x \ast z)^k = (0 \ast x^k) \ast (0 \ast z^k) \in A \text{ or } x \ast z \in [A; k].$$

Hence $[A; k]$ is a quasi-associative ideal of $X$. □

Definition 3 (Hoo [1]). An ideal $A$ of $X$ is said to be implicative if whenever $(x \ast y) \ast z \in A$ and $y \ast z \in A$ then $x \ast z \in A$.

Theorem 2. If $A$ is an implicative ideal of $X$, then the $k$-nil radical of $A$ is also an implicative ideal of $X$ for every positive integer $k$.

Proof. We note from Lemma 4 that $[A; k]$ is an ideal of $X$. Let $x, y, z \in X$ be such that $(x \ast y) \ast z \in [A; k]$ and $y \ast z \in [A; k]$. Then

$$0 \ast ((x \ast y) \ast z)^k = (0 \ast (x \ast y)^k) \ast (0 \ast z^k)$$

$$= ((0 \ast x^k) \ast (0 \ast y^k)) \ast (0 \ast z^k) \in A$$

and $0 \ast (y \ast z)^k = (0 \ast y^k) \ast (0 \ast z^k) \in A$. Since $A$ is an implicative ideal, it follows that $0 \ast (x \ast z)^k = (0 \ast x^k) \ast (0 \ast z^k) \in A$ or equivalently $x \ast z \in [A; k]$. Hence $[A; k]$ is an implicative ideal of $X$. □
THEOREM 3. Let \( f : X \rightarrow Y \) be a homomorphism of BCI-algebras and let \( A \) be a subset of \( X \). Then \( f([A; k]) \subseteq [f(A); k] \) for every positive integer \( k \).

PROOF. Let \( y \in f([A; k]) \). Then there exists \( x \in [A; k] \) such that \( f(x) = y \). It follows that
\[
0 \ast y^k = f(0) \ast (f(x))^k = f(0 \ast x^k) \in f(A)
\]
so that \( y \in [f(A); k] \), ending the proof. \( \Box \)

THEOREM 4. Let \( f : X \rightarrow Y \) be a homomorphism of BCI-algebras and let \( A \) be a subalgebra of \( Y \). Then \( f^{-1}([A; k]) \) is a subalgebra of \( X \) containing \([f^{-1}(A); k] \) for every positive integer \( k \).

PROOF. Let \( x, y \in f^{-1}([A; k]) \). Then \( f(x), f(y) \in [A; k] \). It follows from Lemma 3 that \( f(x \ast y) = f(x) \ast f(y) \in [A; k] \) or equivalently \( x \ast y \in f^{-1}([A; k]) \), which shows \( f^{-1}([A; k]) \) is a subalgebra of \( X \). To prove that \([f^{-1}(A); k] \subseteq f^{-1}([A; k]) \), let \( x \in [f^{-1}(A); k] \). Then \( 0 \ast x^k \in f^{-1}(A) \) which implies that \( 0 \ast (f(x))^k = f(0) \ast f(x^k) = f(0 \ast x^k) \in A \). Thus \( f(x) \in [A; k] \) or equivalently \( x \in f^{-1}([A; k]) \). This completes the proof. \( \Box \)

Note that the inverse image of an ideal under a BCI-homomorphism is an ideal. Hence we have the following theorem

THEOREM 5. Let \( f : X \rightarrow Y \) be a homomorphism of BCI-algebras. If \( A \) is an ideal of \( Y \), then \( f^{-1}([A; k]) \) is an ideal of \( X \) containing \([f^{-1}(A); k] \) for every positive integer \( k \).

Let \( X \) and \( Y \) be BCI-algebras. We define \( \ast \) on \( X \times Y \) by
\[
(x, y) \ast (u, v) = (x \ast u, y \ast v) \quad \text{for every } (x, y), (u, v) \in X \times Y.
\]
Then clearly \((X \times Y; \ast, (0, 0)) \) is a BCI-algebra.

Next we shall define the quotient algebra \( X/[A; k] \) of \( X \) by \([A; k] \). Let \( A \) be a closed ideal of \( X \) and let \( k \) be a positive integer. We define a relation \( \sim \) on \( X \) by \( x \sim y \) if and only if \( x \ast y \in [A; k] \) for every \( x, y \in X \).
Then $\sim$ is an equivalence relation on $X$. In fact, by using Lemma 1(i), we have

$$0 \ast (x \ast x)^k = (0 \ast x^k) \ast (0 \ast x^k) = 0 \in A \text{ for every } x \in X,$$

which implies that $x \ast x \in [A; k]$ or equivalently $x \sim x$.

If $x \sim y$, then $x \ast y \in [A; k]$ and hence, from Lemma 2(ii), $y \ast x \in [A; k]$. Hence $y \sim x$.

Assume that $x \sim y$ and $y \sim z$. Then $x \ast y \in [A; k]$ and $y \ast z \in [A; k]$. Hence $(x \ast z) \ast (x \ast y) \leq y \ast z$ implies $x \ast z \in [A; k]$, since $[A; k]$ is an ideal. Therefore $x \sim z$. Consequently $\sim$ is an equivalence relation on $X$.

Denote by $C_x$ the equivalence class containing $x$, and by $X/[A; k]$ the set of all equivalence classes. We claim that $C_0 = [A; k]$. Let $x \in [A; k]$. Then

$$0 \ast (x \ast 0)^k = (0 \ast x^k) \ast (0 \ast 0^k) = (0 \ast x^k) \ast 0 = 0 \ast x^k \in A,$$

which implies that $x \ast 0 \in [A; k]$, i.e., $x \sim 0$. Hence $x \in C_0$. Conversely, let $x \in C_0$. Then $x \sim 0$ or $x \ast 0 \in [A; k]$. It follows that $0 \ast x^k = 0 \ast (x \ast 0)^k \in A$ so that $x \in [A; k]$. Hence $C_0 = [A; k]$.

Now we shall define a binary operation $\ast$ on $X/[A; k]$. For any $C_x, C_y \in X/[A; k]$, $C_x \ast C_y$ is defined as the class containing $x \ast y$. We can easily check that $(X/[A; k]; \ast, C_0)$ is a BCI-algebra which is called the quotient algebra of $X$ by $[A; k]$.

**Lemma 5** (Jun et al. [4, Proposition 5]). Let $X$ and $Y$ be BCI-algebras. For any $(x, y) \in X \times Y$, we have

$$(0, 0) \ast (x, y)^k = (0 \ast x^k, 0 \ast y^k)$$

for every positive integer $k$.

**Theorem 6.** Let $A$ and $B$ be subsets of BCI-algebras $X$ and $Y$, respectively and $k$ a positive integer. Then

(i) $[A; k] \times [B; k] = [A \times B; k]$.

(ii) if $A$ and $B$ are closed ideals of $X$ and $Y$ respectively, then

$$X/[A; k] \times Y/[B; k] \cong X \times Y/[A \times B; k].$$
PROOF. (i) We have that
\[
[A \times B; k] = \{(x, y) \in X \times Y|(0, 0) \ast (x, y)^k \in A \times B\}
\]
\[
= \{(x, y) \in X \times Y|(0 \ast x^k, 0 \ast y^k) \in A \times B\}
\]
\[
= \{(x, y) \in X \times Y|0 \ast x^k \in A, 0 \ast y^k \in B\}
\]
\[
= \{x \in X|0 \ast x^k \in A\} \times \{y \in Y|0 \ast y^k \in B\}
\]
\[
= [A; k] \times [B; k],
\]
proving (i).

(ii) We note that $[A; k] \times [B; k]$ is an ideal of $X \times Y$ whenever $A$ and $B$ are ideals of $X$ and $Y$, respectively. Consider the natural homomorphisms
\[
\pi_X : X \to X/[A; k], \quad x \mapsto C_x,
\]
\[
\pi_Y : Y \to Y/[B; k], \quad y \mapsto C_y.
\]
Define a mapping $f : X \times Y \to X/[A; k] \times Y/[B; k]$ by $f(x, y) = (C_x, C_y)$ for every $(x, y) \in X \times Y$. Then clearly $f$ is well-defined onto homomorphism. Moreover
\[
Ker f = \{(x, y) \in X \times Y|f(x, y) = ([A; k], [B; k])\}
\]
\[
= \{(x, y) \in X \times Y|(C_x, C_y) = ([A; k], [B; k])\}
\]
\[
= \{(x, y) \in X \times Y|C_x = [A; k], C_y = [B; k]\}
\]
\[
= \{(x, y) \in X \times Y|0 \ast x^k \in A, 0 \ast y^k \in B\}
\]
\[
= \{x \in X|0 \ast x^k \in A\} \times \{y \in Y|0 \ast y^k \in B\}
\]
\[
= [A; k] \times [B; k].
\]
By the first isomorphism theorem, we have
\[
X \times Y/[A \times B; k] \cong X/[A; k] \times Y/[B; k].
\]
This completes the proof. \qed
References


Y. B. Jun,
Department of Mathematics and Research Institute of Natural Science
Gyeongsang National University
Chinju 660-701, Korea
E-mail: ybjun@nongae.gsu.ac.kr

S. M. Hong and E. H. Roh
Department of Mathematics and Research Institute of Natural Science
Gyeongsang National University
Chinju 660-701, Korea