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STRONG LAWS FOR ARRAYS OF RANDOM VARIABLES

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Abstract. In this paper, we obtain an analogue of law of the iterated logarithm for an array of independent, but not necessarily identically distributed, random variables under some moment conditions of the array.

1. Introduction

Let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of independent, but not necessarily identically distributed, random variables with $EX_{ni} = 0$ and $EX_{ni}^2 < \infty$ for $1 \le i \le n$ and $n \ge 1$. Define $S_n = \sum_{i=1}^n X_{ni}$ and $s_n^2 = \sum_{i=1}^n EX_{ni}^2$. In the case of i.i.d. Bernoulli random variables $\{X_{ni}\}$ with $P(X_{11} = \pm 1) = 1/2$, Hu [2] showed that

(1)
$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log s_n^2}} = 1 \text{ a.s.} \quad \text{and} \quad \liminf_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log s_n^2}} = -1 \text{ a.s.}$$

Hu and Weber [3] proved the result (1) under the weaker condition that $\{X_{ni}\}$ is an array of i.i.d. random variables with $EX_{11} = 0$ and $E|X_{11}|^4 < \infty$. Qi [4] proved that for an array of i.i.d. random variables $\{X_{ni}\}$, (1) holds if and only if $EX_{11} = 0$ and $E|X_{11}|^4(\log^+ |X_{11}|)^{-2} < \infty$. Note that from (1) it follows that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log\log s_n^2}} = \infty \text{ a.s.},$$

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so Hartman and Wintner's law of the iterated logarithm cannot hold for arrays.

When the array $\{X_{ni}\}$ is independent, but not necessarily identically distributed, random variables, some law of the iterated logarithm type theorems can be found in Baxter [1], Rosalsky [5], Rosalsky and Teicher [6], Sung [7], and Teicher [8]. Recently, Sung [7] proved that Kolmogorov's law of the iterated logarithm does not hold for arrays.

In this paper, we obtain an analogue of law of the iterated logarithm for an array $\{X_{ni}\}$ of independent, but not necessarily identically distributed, random variables with $s_n^2 \sim n$ and $\sup_{n,i} EX_{ni}^4 < \infty$.

2. Main results

To prove our main theorem, we need the following lemma.

LEMMA 1 (SUNG [7]). *Let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of independent random variables with $EX_{ni} = 0$ for $1 \le i \le n$ and $n \ge 1$. Set $S_n = \sum_{i=1}^n X_{ni}$ and $s_n^2 = \sum_{i=1}^n EX_{ni}^2$. Let $\{k_n\}$ be a sequence of positive constants such that $k_n \to 0$ as $n \to \infty$. Suppose that the following conditions hold.*

(i) *$s_n^2 = n$ for $n \ge 1$.*
(ii) *$|X_{ni}| \le k_n\sqrt{n}/\sqrt{\log n}$ a.s. for $1 \le i \le n$ and $n \ge 1$.*

Then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log n}} = 1 \text{ a.s.}$$

The following theorem states that the condition (ii) of Lemma 1 can be replaced by simple moment conditions of the array.

THEOREM 2. *Let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of independent random variables with $EX_{ni} = 0$ for $1 \le i \le n$ and $n \ge 1$. Set $S_n = \sum_{i=1}^n X_{ni}$ and $s_n^2 = \sum_{i=1}^n EX_{ni}^2$. Suppose that $s_n^2 \sim n$ and $\sup_{n,i} EX_{ni}^4 < \infty$. Then*

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log n}} = 1 \text{ a.s.}$$

Proof. For $2 < q < 4$, define $X'_{ni} = X_{ni} I(|X_{ni}| \le n^{1/q})$. Observe that, by the Cauchy-Schwarz inequality and the Markov inequality, for all $1 \le i \le n$ and $n \ge 1$

$$\begin{aligned} EX_{ni}^2 I(|X_{ni}| > n^{1/q}) &\le (EX_{ni}^4)^{1/2} (P(|X_{ni} > n^{1/q}))^{1/2} \\ &\le (EX_{ni}^4)^{1/2} (\frac{EX_{ni}^4}{n^{4/q}})^{1/2} \qquad (2) \\ &\le \frac{\sup_{n,i} EX_{ni}^4}{n^{2/q}}. \end{aligned}$$

Since $\sup_{n,i} EX_{ni}^4 < \infty$, it follows by (2) that $\sum_{i=1}^n EX_{ni}^2 I(|X_{ni}| > n^{1/q})/n \to 0$ and $\sum_{i=1}^n (EX_{ni} I(|X_{ni}| > n^{1/q}))^2/n \to 0$ as $n \to \infty$. Thus $\sum_{i=1}^n Var(X'_{ni}) \sim n$, since

$$\begin{aligned} n \sim s_n^2 &= \sum_{i=1}^n E(X'_{ni})^2 + \sum_{i=1}^n EX_{ni}^2 I(|X_{ni}| > n^{1/q}) \\ &= \sum_{i=1}^n Var(X'_{ni}) + \sum_{i=1}^n (EX'_{ni})^2 + \sum_{i=1}^n EX_{ni}^2 I(|X_{ni}| > n^{1/q}) \\ &= \sum_{i=1}^n Var(X'_{ni}) + \sum_{i=1}^n (EX_{ni} I(|X_{ni}| > n^{1/q}))^2 \\ &\qquad + \sum_{i=1}^n EX_{ni}^2 I(|X_{ni}| > n^{1/q}). \end{aligned}$$

Let ${s'_n}^2 = \sum_{i=1}^n Var(X'_{ni})$ and $Y_{ni} = \sqrt{n}(X'_{ni} - EX'_{ni})/s'_n$. Then $EY_{ni} = 0, \sum_{i=1}^n EY_{ni}^2 = n$, and $\max_{1 \le i \le n} |Y_{ni}| \le 2\sqrt{n} n^{1/q}/s'_n \sim 2n^{1/q}$ $= o(\sqrt{n}/\sqrt{\log n})$. Thus, $\{Y_{ni},\ 1 \le i \le n,\ n \ge 1\}$ satisfies the conditions of Lemma 1, and so

$$\limsup_{n \to \infty} \frac{\sum_{i=1}^n Y_{ni}}{\sqrt{2n \log n}} = 1 \text{ a.s.}$$

Since $s'_n \sim \sqrt{n}$,

$$\limsup_{n \to \infty} \frac{\sum_{i=1}^n (X'_{ni} - EX'_{ni})}{\sqrt{2n \log n}} = 1 \text{ a.s.} \qquad (3)$$

Using (2) and $EX_{ni} = 0$ for all n and i, it follows that

$$\begin{aligned}\frac{|\sum_{i=1}^n EX'_{ni}|}{\sqrt{n\log n}} &= \frac{|\sum_{i=1}^n EX_{ni}I(|X_{ni}| > n^{1/q})|}{\sqrt{n\log n}} \\ &\le \frac{\sum_{i=1}^n EX_{ni}^2 I(|X_{ni}| > n^{1/q})}{n^{1/q}\sqrt{n\log n}} \\ &\le \frac{\sup_{n,i} EX_{ni}^4}{(\log n)^{1/2} n^{-1/2+1/q+2/q}} \to 0,\end{aligned}$$

since $q < 4$. Combining this and (3) gives

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n X'_{ni}}{\sqrt{2n\log n}} = 1 \text{ a.s.}$$

To finish the proof, it suffices to show that

$$\frac{\sum_{i=1}^n (X_{ni} - X'_{ni})}{\sqrt{n\log n}} \to 0 \text{ a.s.},$$

which, by the Borel-Cantelli lemma, is equivalent to the fact for every $\epsilon > 0$

$$\sum_{n=1}^\infty P\left(\frac{|\sum_{i=1}^n (X_{ni} - EX'_{ni})|}{\sqrt{n\log n}} > \epsilon\right) < \infty. \tag{4}$$

Fix $\epsilon > 0$. Let $X''_{ni} = X_{ni}I(|X_{ni}| > \epsilon\sqrt{n\log n}/N)$ and $X'''_{ni} = X_{ni}I(n^{1/q} < |X_{ni}| \le \epsilon\sqrt{n\log n}/N)$, where N is a positive integer to be chosen later. Then $X_{ni} - X'_{ni} = X''_{ni} + X'''_{ni}$. Hence, to prove (4), it is enough to show that

$$\sum_{n=1}^\infty P\left(\frac{|\sum_{i=1}^n X''_{ni}|}{\sqrt{n\log n}} > \epsilon\right) < \infty \tag{5}$$

and

$$\sum_{n=1}^\infty P\left(\frac{|\sum_{i=1}^n X'''_{ni}|}{\sqrt{n\log n}} > \epsilon\right) < \infty. \tag{6}$$

From the definition of X''_{ni} and the Markov inequality, we get

$$\begin{aligned} P\left(\frac{|\sum_{i=1}^n X''_{ni}|}{\sqrt{n\log n}} > \epsilon\right) &\le P\left(\cup_{i=1}^n \left(|X_{ni}| > \frac{\epsilon\sqrt{n\log n}}{N}\right)\right) \\ &\le \sum_{i=1}^n P\left(|X_{ni}| > \frac{\epsilon\sqrt{n\log n}}{N}\right) \\ &\le \sup_{n,i} EX_{ni}^4 \frac{N^4}{\epsilon^4 n(\log n)^2}, \end{aligned}$$

and so (5) holds.

Since $|X'''_{ni}| \le \epsilon\sqrt{n\log n}/N$, $\sum_{i=1}^n |X'''_{ni}|/\sqrt{n\log n} > \epsilon$ implies that there are at least N nonzeros X'''_{ni} for $i = 1, \cdots, n$. Hence, by the Markov inequality, we have

$$\begin{aligned} P\left(\frac{|\sum_{i=1}^n X'''_{ni}|}{\sqrt{n\log n}} > \epsilon\right) &\le P\left(\frac{\sum_{i=1}^n |X'''_{ni}|}{\sqrt{n\log n}} > \epsilon\right) \\ &\le \sum_{k_1<\cdots<k_N} P(|X'''_{nk_1}| \ne 0, \cdots, |X'''_{nk_N}| \ne 0) \\ &\le \sum_{k_1<\cdots<k_N} P(|X_{nk_1}| > n^{1/q}, \cdots, |X_{nk_N}| > n^{1/q}) \\ &= \sum_{k_1<\cdots<k_N} P(|X_{nk_1}| > n^{1/q}) \cdots P(|X_{nk_N}| > n^{1/q}) \\ &\le \sum_{k_1<\cdots<k_N} \frac{EX_{nk_1}^4 \cdots EX_{nk_N}^4}{n^{4N/q}} \\ &\le \binom{n}{N} \frac{\sup_{n,i}(EX_{ni}^4)^N}{n^{4N/q}} \\ &\le \frac{\sup_{n,i}(EX_{ni}^4)^N}{n^{(4/q-1)N}}, \end{aligned}$$

where the summation $\sum_{k_1<\cdots<k_N}$ is taken for all N-tuples $(k_1, \cdots, k_N)$ with $k_1 < \cdots < k_N$ and $k_i = 1, \cdots, n$ for each i. Thus, choosing N such that $(4/q - 1)N > 1$, (6) holds. □

COROLLARY 3. *Let* $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ *be as in* Theorem 2. *Assume that* $0 < a \le EX_{ni}^2$ *for all* n *and* i, *and* $\sup_{n,i} EX_{ni}^4 < \infty$. *Then*

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n X_{ni}}{\sqrt{2s_n^2 \log n}} = 1 \text{ a.s.}$$

Proof. Let $Y_{ni} = \sqrt{n} X_{ni}/s_n$ for all n and i. Then $\{Y_{ni}\}$ satisfies the conditions of Theorem 2. Thus, the result follows from Theorem 2. □

REMARK. Hu and Weber ([3], Theorem 3) proved Corollary 3 under the stronger condition that for each n, $X_{n1}, \cdots, X_{nn}$ are identically distributed and $\sup_{n,i} E|X_{ni}|^{4+\delta} < \infty$ for some $\delta > 0$.

The following example shows that Corollary 3 does not hold if the condition $0 < a \le EX_{ni}^2$ in Corollary 3 is removed

EXAMPLE. Let $\{Y_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of independent Bernoulli random variables with

$$P(Y_{ni} = 1) = \frac{\log_2 n}{n} = 1 - P(Y_{ni} = 0) \text{ for } 1 \le i \le n \text{ and } n \ge 1,$$

where $\log_2 t = \log\log t$ for $t > 1$. Let $X_{ni} = Y_{ni} - \frac{\log_2 n}{n}$ for all n and i. Then $|X_{ni}| \le 1$, $EX_{ni} = 0$, and $EX_{ni}^2 = \frac{\log_2 n}{n}(1 - \frac{\log_2 n}{n})$. Thus $\{X_{ni}\}$ satisfies the conditions of Corollary 3, except for the condition $0 < a \le EX_{ni}^2$. But, it follows from Example 1 in Rosalsky ([5], p. 388) that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\sum_{i=1}^n X_{ni}}{\sqrt{2s_n^2 \log n}} &= \limsup_{n\to\infty} \frac{\sum_{i=1}^n (Y_{ni} - \log_2 n/n)}{\sqrt{2\log_2 n(1 - \log_2 n/n)\log n}} \\
&= \limsup_{n\to\infty} \frac{\sum_{i=1}^n Y_{ni}}{\sqrt{2\log_2 n \log n}} \\
&= \limsup_{n\to\infty} \frac{\sum_{i=1}^n Y_{ni}\log_2 n}{\log n} \frac{\log n}{\log_2 n\sqrt{2\log_2 n\log n}} \\
&= \infty \text{ a.s.}
\end{aligned}$$

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