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WAVELET REPRESENTATION OF DERIVATIVE OPERATORS: ALTERNATIVE DERIVATION

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ABSTRACT. The original work for representing derivative operators in the wavelet bases was done by Beylkin in [2]. In this paper we present an alternative and easier derivation.

1. Introduction and preliminaries

In [3], Daubechies introduces compactly supported wavelets which are very useful in numerical analysis [1]. In [2], G. Beylkin introduces representations of operators, such as the operator of differentiation, *etc.*, in orthonormal bases of compactly supported wavelets. In this paper, we are mainly focus on the derivative operators.

The representation of derivative operators is completely determined by the coefficients r_l in the subspace V_0 of $L^2(\mathbb{R})$ (see [2]). In this paper, we give an alternative and easier derivation of the representation of derivative operators, and connection coefficients r_l, which are used for representing derivative operators and are exactly the same coefficients r_l in [2].

In this section, we briefly review the fundamentals of wavelet theory. The standard references for wavelets are [3,4].

A *multiresolution analysis* of $L^2(\mathbb{R})$ is a sequence of closed subspaces $\{V_k\}_{k\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ with the following properties:

(i) $V_k \subset V_{k+1}$,

(ii) $\bigcup_{k\in\mathbb{Z}} V_k$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{k\in\mathbb{Z}} V_k = \{0\}$,

(iii) $f(x) \in V_k \iff f(2x) \in V_{k+1}$,

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(iv) $f(x) \in V_0 \iff f(x-j) \in V_0$ for each $j \in \mathbb{Z}$,

(v) there exists a function $\phi(x) \in V_0$, called the *scaling function,* such that $\{\phi(x-j)\}_{j\in\mathbb{Z}}$ forms a Riesz basis of V_0.

If the family of the integer translates $\{\phi(x-j)\}_{j\in\mathbb{Z}}$ of a scaling function ϕ forms an orthogonal basis of V_0, then we call ϕ an *orthogonal scaling function.*

We denote the scaled translates of ϕ by

$$\phi_j^k(x) = 2^{k/2}\phi(2^k x - j).$$

Then, for fixed $k \in \mathbb{Z}$, the $\{\phi_j^k\}_{j\in\mathbb{Z}}$ forms a Riesz basis of V_k.

We assume that the basis functions are normalized, i.e.,

$$\int_{-\infty}^{\infty} \phi(x)\, dx = 1.$$

Let P_k be the orthogonal projection from $L^2(\mathbb{R})$ onto V_k; that is, $P_k f$ is the best $L^2(\mathbb{R})$ approximation to $f \in L^2(\mathbb{R})$ from V_k. If $\{\phi_j^k\}_{j\in\mathbb{Z}}$ is an orthonormal basis of V_k, then we obtain the orthogonal projector P_k from $L^2(\mathbb{R})$ onto V_k as

$$P_k f(x) = \sum_{j=-\infty}^{\infty} \langle f, \phi_j^k\rangle \phi_j^k(x),$$

where $\langle\ ,\ \rangle$ is the standard real inner product of two functions in $L^2(\mathbb{R})$. We call $P_k f$ the *wavelet approximation* to f at the resolution $h = 2^{-k}$, and $\langle f, \phi_j^k\rangle$ the *wavelet coefficients* of f.

2. The representation of operators in the wavelet bases

Let V_k and V_l be two subspaces of $L^2(\mathbb{R})$ with respective bases $\{\phi_i^k\}_{i\in\mathbb{Z}}$ and $\{\phi_i^l\}_{i\in\mathbb{Z}}$.

Let $\mathbf{T}$ be a linear operator from V_k to V_l. Then the image of ϕ_i^k can be expressed as a linear combination of $\{\phi_i^l\}$:

$$\mathbf{T}\phi_j^k = \sum_i t_{ij}\phi_i^l. \tag{2.1}$$

The coefficient t_{ij} is the ith component of $\mathbf{T}\phi_j^k$ in the basis $\{\phi_i^l\}$, and $T = (t_{ij})$ is the matrix representation of $\mathbf{T}$ in the bases $\{\phi_i^k\}$ and $\{\phi_i^l\}$. If $\{\phi_i^k\}$ is an orthogonal basis for V_k, then $t_{ij} = \langle \mathbf{T}\phi_j^k, \phi_i^l \rangle$.

More generally, $T = (t_{ij})$ with $t_{ij} = \langle \mathbf{T}\phi_j^k, \phi_i^l \rangle$ represents $P_l \mathbf{T} P_k$, where $\mathbf{T} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. If $\{\phi_i^k\}$ and $\{\phi_i^l\}$ are orthonormal bases for V_k and V_l, respectively, then

$$P_l \mathbf{T} P_k \phi_j^k = \sum_i t_{ij} \phi_i^l = \sum_i \langle P_l \mathbf{T} P_k \phi_j^k, \phi_i^l \rangle \phi_i^l. \tag{2.2}$$

In particular, if $\mathbf{T}$ is the n^{th} order derivative operator, then

$$t_{ij} = \int_{-\infty}^{\infty} \phi_i^k(x) \frac{d^n}{dx^n} \phi_j^k(x)\, dx \tag{2.3}$$

and $T = (t_{ij})$ represents $P_k \mathbf{T} P_k$.

3. The scaling function quto-correlation

Assume that the scaling function ϕ satisfies the *dilation equation*

$$\phi(x) = \sqrt{2} \sum_j h_j\, \phi(2x - j) \tag{3.1}$$

and the h coefficients in the dilation equation satisfy

$$\sum_j h_j = \sqrt{2} \qquad \text{and} \qquad \sum_j (-1)^j h_j = 0.$$

Define the *auto-correlation* function r by

$$r(x) = \int_{-\infty}^{\infty} \phi(y - x)\, \phi(y)\, dy. \tag{3.2}$$

Note that $r(x)$ is an even function. By substituting the dilation equation into (3.2), we obtain

$$r(x) = \sqrt{2} \sum_n a_n r(2x - n) \tag{3.3}$$

where

$$a_n = \frac{1}{\sqrt{2}} \sum_j h_j h_{j+n} \qquad \text{and} \qquad a_{-n} = a_n. \tag{3.4}$$

Note that if $\{\phi_i(x)\}_{i\in\mathbb{Z}}$ is an orthonormal basis for V_0, then

$$a_{2l} = \frac{1}{\sqrt{2}} \sum_j h_j h_{j+2l} = \frac{1}{\sqrt{2}} \delta_{0l}.$$

where δ is the Kronecker delta function. Since $\sum_n a_n = \sqrt{2}$ and $\sum_n (-1)^n a_n = 0$, $r(x)$ is another scaling function if ϕ has compact support.

If subscripts of h_j run from l_1 to l_2, then ϕ has support in $[l_1, l_2]$. Let $L = l_2 - l_1$. Then $L+1$ is the number of coefficients h_j and L is the length of the support of ϕ. The subscripts of a_n run from $-L$ to L. So $r(x)$ has support in $[-L, L]$. The number of coefficients h_j, $L+1$, is related to the number of vanishing moments M for the wavelet ψ. For the Daubechies wavelets, $L+1 = 2M$. If additional conditions are imposed, then the relation might be different, but $L+1$ is always even.

To calculate point values of $r(x)$, we begin for $x \in \mathbb{Z}$. For $l \in \mathbb{Z}$, we denote $r(l)$ by r_l, i.e.,

$$r_l := r(l) = \int_{-\infty}^{\infty} \phi(x-l)\, \phi(x)\, dx. \tag{3.5}$$

The r_l are called the *connection coefficients.* The fact that

$$r_l = \sqrt{2} \sum_{n=-L}^{L} a_n r_{2l-n} = \sqrt{2} \sum_{m=2l-L}^{2l+L} a_{2l-m} r_m$$

leads to an eigenvalue problem of size $2L+1$

$$\vec{r} = A\vec{r}, \tag{3.6}$$

where

$$A = (\sqrt{2}\, a_{2i-j})_{-L \le i, 2i-j \le L},$$

$$\vec{r} = (r_{-L}, \cdots, r_L)^T.$$

In fact,

$$(3.7) \qquad A = \sqrt{2} \begin{pmatrix} a_{-L} & & & & & & \\ a_{-L+2} & a_{-L+1} & a_{-L} & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & a_L & a_{L-1} & a_{L-2} \\ & & & & & & a_L \end{pmatrix}$$

Since all column sums of A are either

$$\sqrt{2} \sum_k a_{2k} = 1$$

or

$$\sqrt{2} \sum_k a_{2k+1} = 1,$$

$(1, 1, \cdots, 1)$ is a left eigenvector with eigenvalue 1. Hence, a right eigenvector $\vec{r}$ for eigenvalue 1 exists.

If $\{\phi_i(x)\}_{i \in \mathbb{Z}}$ is orthonormal basis for V_0, then the eigenvector $\vec{r}$ for eigenvalue 1 is always $(0, \cdots, 0, 1, 0, \cdots, 0)^T$. Because $\sqrt{2} a_{2l} = \delta_{0l}$ this implies that

$$(A - I)\vec{r} = \sqrt{2}(0, \cdots, a_{-2}, a_0, a_2, \cdots, 0)^T - (0, \cdots, 0, 1, 0, \cdots, 0)^T = \vec{0}.$$

In this case, $r(l) = \int_{-\infty}^{\infty} \phi(x - l)\phi(x)\, dx = \delta_{0l}$.

Therefore, we normalize r_l with

$$(3.8) \qquad \sum_l r_l = 1.$$

The first row of (3.6) is $r_{-L} = \sqrt{2} a_{-L}\, r_{-L}$. If $\sqrt{2} a_{-L} \neq 1$, then $r_{-L} = 0$ and $r_L = 0$ also, since $a_{-L} = a_L$. This condition is usually satisfied, for example, if ϕ is a Daubechies scaling function. In this case, it suffices to solve an eigenvalue problem of size $2L-1$

$$\vec{r} = A\vec{r}, \tag{3.9}$$

where

$$A = (\sqrt{2}\, a_{2i-j})_{-L+1\le i, 2i-j \le L-1}$$

$$\vec{r} = (r_{-L+1}, \cdots, r_{L-1})^T.$$

In fact,
(3.10)

$$A = \sqrt{2}\begin{pmatrix} a_{-L+1} & a_{-L} & & & & & & \\ a_{-L+3} & a_{-L+2} & a_{-L+1} & a_{-L} & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & a_L & a_{L-1} & a_{L-2} & a_{L-3} \\ & & & & & a_L & a_{L-1} \end{pmatrix}.$$

Hence, we have the following theorem.

THEOREM 3.1. *If ϕ is an orthonormal scaling function with compact support, and if $\sqrt{2}a_{-L} \neq 1$, then the connection coefficients r_l can be determined uniquely by solving* (3.9) *and* (3.8).

The values of $r(x)$ at all dyadic points x can be calculated from the dilation equation. This is identical to the procedure for finding point values of ϕ ([4,6]).

Now we are ready to represent derivatives.

4. The n^{th} order derivative operator in the wavelet bases

Let $\mathbf{T}$ be the n^{th} order derivative operator. For any positive integer n, define the function $r^{(n)}(x)$ by

$$r^{(n)}(x) = \int_{-\infty}^{\infty} \phi(y-x)\, \frac{d^n}{dy^n}\phi(y)\, dy. \tag{4.1}$$

The relationship between the n^{th} derivative of the scaling function autocorrelation and $r^{(n)}(x)$ is

$$\frac{d^n}{dx^n}r(x) = (-1)^n r^{(n)}(-x) = r^{(n)}(x). \tag{4.2}$$

For $l \in \mathbb{Z}$,

$$r_l^{(n)} := r^{(n)}(l) = \int_{-\infty}^{\infty} \phi(x-l)\, \frac{d^n}{dx^n}\phi(x)\, dx. \tag{4.3}$$

Recall the matrix representation of $P_k \mathbf{T} P_k$ in section 1, where P_k is the orthogonal projection on the subspace V_k of $L^2(\mathbb{R})$. We can express the entries t_{ij} of the matrix representation of $P_k \mathbf{T} P_k$ with $r_l^{(n)}$:

$$t_{ij} = \int_{-\infty}^{\infty} (\phi_j^k)^{(n)}(x)\, \phi_i^k(x)\, dx = 2^{nk} r_{i-j}^{(n)} = \frac{1}{h^n} r_{i-j}^{(n)}, \tag{4.4}$$

where $h = 2^{-k}$. By repeated integration by parts in (4.3), we obtain the odd and even properties for $r_l^{(n)}$:

$$r_{-l}^{(n)} = \begin{cases} -r_l^{(n)} & \text{for odd } n, \\ r_l^{(n)} & \text{for even } n. \end{cases} \tag{4.5}$$

Now

$$r^{(n)}(x) = \sqrt{2} \sum_n a_n 2^n r^{(n)}(2x-n).$$

The fact that

$$r_l^{(n)} = 2^n \sqrt{2} \sum_{n=-L}^{L} a_n r_{2l-n}^{(n)} = 2^n \sqrt{2} \sum_{m=2l-L}^{2l+L} a_{2l-m} r_m^{(n)}$$

leads to an eigenvalue problem

$$\vec{r}^{(n)} = 2^n A \vec{r}^{(n)}, \tag{4.6}$$

where A is the same matrix as (3.7)

$$A = (\sqrt{2}\, a_{2i-j})_{-L \le i, 2i-j \le L},$$

$$\vec{r}^{(n)} = (r_{-L}^{(n)}, \dots, r_{L}^{(n)})^T.$$

If A has an eigenvalue 2^{-n}, then the eigenvector $\vec{r}^{(n)}$ exists. For a unique solution, we need the correct normalization.

If $\{\phi_i^k(x)\}_{i \in \mathbb{Z}}$ is orthonormal basis for V_k, then

$$\sum_l l^n r_l^{(n)} = (-1)^n n! \tag{4.7}$$

(see Lemma 5.10 in [5]). Relation (4.7) is used to normalize the eigenvectors of A correctly.

The first row of (4.6) is $r_{-L}^{(n)} = 2^n \sqrt{2} a_{-L} r_{-L}^{(n)}$. If $2^n \sqrt{2} a_{-L} \neq 1$, then $r_{-L}^{(n)} = 0$ and $r_{L}^{(n)} = 0$ also, since $a_{-L} = a_L$. This condition is usually satisfied, for example, if ϕ is a Daubechies scaling function. In this case, it suffices to solve an eigenvalue problem of size $2L - 1$

$$\vec{r}^{(n)} = 2^n A \vec{r}^{(n)}, \tag{4.8}$$

where A is the same as (3.10)

$$A = (\sqrt{2}\, a_{2i-j})_{-L+1 \le i, 2i-j \le L-1}$$

$$\vec{r}^{(n)} = (r_{-L+1}^{(n)}, \dots, r_{L-1}^{(n)})^T.$$

Hence, we have the following theorem, the counterpart of Propositions 1 and 2 in [2].

THEOREM 4.1. *If ϕ is an orthonormal scaling function with compact support, if the matrix A has an eigenvalue 2^{-n}, and if $2^n \sqrt{2} a_{-L} \neq 1$, then the coefficients $r_l^{(n)}$, used for representing nth derivative operators, can be determined uniquely by solving (4.8) and (4.7).*

5. Examples

We compute the nonzero connection coefficients used for representing derivative operators. The results in this section derived by our approach are exactly the same as the results derived by Beylkin in [2]. Note that a_j's here are not the same a_j's in [2]. For each j, a_j here is $1/2\sqrt{2}$ times the a_j in [2].

EXAMPLE 5.1. Let ϕ be the Daubechies scaling function with 2 vanishing moments for ψ. The length of the support of ϕ is $L = 3$. $a_0 = 1/\sqrt{2}$, $a_{\pm1} = 9/16\sqrt{2}$, and $a_{\pm3} = -1/16\sqrt{2}$. The matrix A of size $2L-1$ is

$$A = \frac{1}{16}\begin{pmatrix} 0 & -1 & & & \\ 16 & 9 & 0 & -1 & \\ 0 & 9 & 16 & 9 & 0 \\ & -1 & 0 & 9 & 16 \\ & & & -1 & 0 \end{pmatrix}.$$

Eigenvalues for A are $1, 1/2, 1/8, \ldots$. We have the first and the third derivatives, but not the second derivative. The values for $r_l^{(n)}$ are as follows:

$$\vec{r}^{(1)} = (-\frac{1}{12}, \frac{2}{3}, 0, -\frac{2}{3}, \frac{1}{12})^T$$

$$\vec{r}^{(3)} = (\frac{1}{2}, -1, 0, 1, -\frac{1}{2})^T.$$

EXAMPLE 5.2. Let ϕ be the Daubechies scaling function with 3 vanishing moments for ψ. The length of the support of ϕ is $L = 5$. $a_0 = 1/\sqrt{2}$, $a_{\pm1} = 75/128\sqrt{2}$, $a_{\pm3} = -25/256\sqrt{2}$, and $a_{\pm5} = 3/256\sqrt{2}$.

The matrix A of size $2L-1$ is

$$A=\frac{1}{256}\begin{pmatrix} 0 & 3 & & & & & & & \\ 0 & -25 & 0 & 3 & & & & & \\ 256 & 150 & 0 & -25 & 0 & 3 & & & \\ 0 & 150 & 256 & 150 & 0 & -25 & 0 & 3 & \\ 0 & -25 & 0 & 150 & 256 & 150 & 0 & -25 & 0 \\ & 3 & 0 & -25 & 0 & 150 & 256 & 150 & 0 \\ & & & 3 & 0 & -25 & 0 & 150 & 256 \\ & & & & & 3 & 0 & -25 & 0 \\ & & & & & & & 3 & 0 \end{pmatrix}.$$

Eigenvalues for A are $1, 1/2, 1/4, 1/8, 1/16, 1/32, \ldots$. We have from the first derivative up to the fifth derivative. Since 2^{-6} is not an eigenvalue of A, there does not exist the 6^{th} derivative. The values for $r_l^{(n)}$ are as follows:

$$\vec{r}^{(1)}=(\frac{1}{2920},\frac{16}{1095},-\frac{53}{365},\frac{272}{365},0,-\frac{272}{365},\frac{53}{365},-\frac{16}{1095},-\frac{1}{2920})^T$$

$$\vec{r}^{(2)}=(\frac{3}{560},\frac{4}{35},-\frac{92}{105},\frac{356}{105},-\frac{295}{56},\frac{356}{105},-\frac{92}{105},\frac{4}{35},\frac{3}{560})^T$$

$$\vec{r}^{(3)}=(-\frac{3}{400},-\frac{2}{25},-\frac{179}{200},-\frac{38}{25},0,\frac{38}{25},\frac{179}{365},\frac{2}{25},\frac{3}{400})^T$$

$$\vec{r}^{(4)}=(\frac{9}{160},\frac{3}{10},-\frac{77}{40},\frac{41}{10},-\frac{81}{16},\frac{41}{10},-\frac{77}{40},\frac{3}{10},\frac{9}{160})^T$$

$$\vec{r}^{(5)}=(\frac{3}{52},\frac{2}{13},-\frac{31}{26},\frac{22}{13},0,-\frac{22}{13},\frac{31}{26},-\frac{2}{13},-\frac{3}{52})^T.$$

Numerical Data. The following was computed using MATLAB for Daubechies wavelet with M vanishing moments. We note, that

$$r_{-l}^{(n)}=\begin{cases} -r_l^{(n)} & \text{for odd } n, \\ r_l^{(n)} & \text{for even } n. \end{cases}$$

- The values of $r_l^{(n)}$ for $M = 4$ are

	$n=1$	$n=2$	$n=3$
$l=0$	0	$-4.1660e+00$	0
$l=1$	$-7.9301e-01$	$2.6421e+00$	$1.8662e+00$
$l=2$	$1.9200e-01$	$-6.9787e-01$	$-1.2160e+00$
$l=3$	$-3.3580e-02$	$1.5097e-01$	$1.9027e-01$
$l=4$	$2.2240e-03$	$-1.0573e-02$	$4.3693e-03$
$l=5$	$1.7221e-04$	$-1.6304e-03$	$-4.5959e-03$
$l=6$	$-8.4085e-07$	$1.5922e-05$	$8.9764e-05$

- The values of $r_l^{(n)}$ for $M = 5$ are

	$n=1$	$n=2$	$n=3$
$l=0$	0	$-3.8350e+00$	0
$l=1$	$-8.2591e-01$	$2.4148e+00$	$2.1240e+00$
$l=2$	$2.2882e-01$	$-6.4950e-01$	$-1.4897e+00$
$l=3$	$-5.3353e-02$	$1.8095e-01$	$3.1853e-01$
$l=4$	$7.4614e-03$	$-2.9908e-02$	$-1.6199e-02$
$l=5$	$-2.3924e-04$	$7.9462e-04$	$-9.1725e-03$
$l=6$	$-5.4047e-05$	$3.6715e-04$	$1.8041e-03$
$l=7$	$-2.5241e-07$	$1.6565e-06$	$-3.7343e-05$
$l=8$	$-2.6960e-10$	$3.5388e-09$	$-1.5955e-07$

• The values of $r_l^{(n)}$ for $M = 6$ are

	$n=1$	$n=2$	$n=3$
$l=0$	0	$-3.6861e+00$	0
$l=1$	$-8.5014e-01$	$2.3119e+00$	$2.3300e+00$
$l=2$	$2.5855e-01$	$-6.3073e-01$	$-1.7317e+00$
$l=3$	$-7.2441e-02$	$2.0491e-01$	$4.5892e-01$
$l=4$	$1.4546e-02$	$-4.9362e-02$	$-5.6286e-02$
$l=5$	$-1.5886e-03$	$6.4781e-03$	$-7.9710e-03$
$l=6$	$4.2969e-06$	$-6.5696e-05$	$4.1778e-03$
$l=7$	$1.2027e-05$	$-5.4363e-05$	$-5.0228e-04$
$l=8$	$4.2069e-07$	$-3.4661e-06$	$1.1303e-06$
$l=9$	$-2.8997e-09$	$2.6300e-08$	$4.8917e-07$
$l=10$	$6.9681e-13$	$-1.2641e-11$	$-4.7024e-10$

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