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THE CURVATURE TENSORS IN THE EINSTEIN'S $^*g$-UNIFIED FIELD THEORY II. THE CONTRACTED SE-CURVATURE TENSORS OF $^*g$-SEX$_n$

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ABSTRACT. Chung and et al. ([2], 1991) introduced a new concept of a manifold, denoted by $^*g$-SEX$_n$, in Einstein's n-dimensional $^*g$-unified field theory. The manifold $^*g$-SEX$_n$ is a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by the unified field tensor $^*g^{\lambda\nu}$ through the SE-connection which is both Einstein and semi-symmetric. In this paper, they proved a necessary and sufficient condition for the unique existence of SE-connection and presented a beautiful and surveyable tensorial representation of the SE-connection in terms of the tensor $^*g^{\lambda\nu}$. Recently, Chung and et al. ([3], 1998) obtained a concise tensorial representation of SE-curvature tensor defined by the SE-connection of $^*g$-SEX$_n$ and proved several identities involving it.

This paper is a direct continuation of [3]. In this paper we derive surveyable tensorial representations of contracted curvature tensors of $^*g$-SEX$_n$ and prove several generalized identities involving them. In particular, the first variation of the generalized Bianchi's identity in $^*g$-SEX$_n$, proved in Theorem (2.10a), has a great deal of useful physical applications.

1. Preliminaries

This paper is a direct continuation of our previous paper [3], which will be denoted by I in the present paper. All considerations in this paper are based on the results and symbolism of I. Whenever necessary,

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these results will be quoted in the text. In this section, we introduce a brief collection of basic concepts, notations, and results of I, which are frequently used in the present paper.

Let X_n be an n-dimensional generalized Riemannian manifold referred to a real coordinate system x^ν. In the Einstein's usual n-dimensional unified field theory, the manifold X_n is endowed with a real non-symmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{1.1a}$$

where

$$det(g_{\lambda\mu}) \neq 0, \quad det(h_{\lambda\mu}) \neq 0. \tag{1.1b}$$

In our n-dimensional ${}^*g$-unified field theory (n-${}^*g$-UFT hereafter), however, *the algebraic structure on* X_n is imposed by the basic real tensor ${}^*g^{\lambda\nu}$, defined by

$$g_{\lambda\mu}{}^*g^{\lambda\nu} \stackrel{\text{def}}{=} g_{\mu\lambda}{}^*g^{\nu\lambda} \stackrel{\text{def}}{=} \delta_\mu^\nu. \tag{1.2a}$$

It may be also decomposed into its symmetric part ${}^*h^{\lambda\nu}$ and skew-symmetric part ${}^*k^{\lambda\nu}$:

$${}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}. \tag{1.2b}$$

Since $det({}^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor ${}^*h_{\lambda\mu}$ by

$${}^*h_{\lambda\mu}{}^*h^{\lambda\nu} \stackrel{\text{def}}{=} \delta_\mu^\nu. \tag{1.2c}$$

In n-${}^*g$-UFT we use both ${}^*h^{\lambda\nu}$ and ${}^*h_{\lambda\mu}$ as tensors for raising and/or lowering indices of all tensors defined in X_n in the usual manner.

On the other hand, *the differential geometric structure on* X_n is imposed by the tensor ${}^*g^{\lambda\nu}$ by means of a connection $\Gamma_{\lambda\mu}^\nu$ defined by a system of equations

$$D_\omega {}^*g^{\lambda\nu} = -2S_{\omega\alpha}{}^\nu \, {}^*g^{\lambda\alpha}. \tag{1.3}$$

Here D_ω denotes the symbol of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma^\nu_{\lambda\mu}$. Under certain conditions the system (1.3) admits a unique solutions $\Gamma^\nu_{\lambda\mu}$. A connection is said to be *Einstein* if it satisfies (1.3).

A connection $\Gamma^\nu_{\lambda\mu}$ is said to be *semi-symmetric* if its torsion tensor $S_{\lambda\mu}{}^\nu$ is of the form

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda} X_{\mu]} \tag{1.4}$$

for an arbitrary vector $X_\lambda \neq 0$, which is not a gradient vector. A connection which is both semi-symmetric and Einstein is called a *SE-connection.* An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by the tensor ${}^*g^{\lambda\nu}$ by means of a SE-connection, is called an *n-dimensional ${}^*g$-SE-manifold.* We denote this manifold by ${}^*g$-SEX$_n$ in our further considerations.

In the present paper, we frequently use the following abbreviations for an arbitrary vector Y_λ, for $p = 1, 2, 3, \cdots$:

$$ {}^{(0)*}k_\lambda{}^\nu = \delta^\nu_\lambda, \quad {}^{(p)*}k_\lambda{}^\nu = {}^*k_\lambda{}^\alpha\, {}^{(p-1)*}k_\alpha{}^\nu \tag{1.5a}$$

$$ {}^{(p)}Y^\nu = {}^{(p-1)*}k^\nu{}_\alpha Y^\alpha = {}^*k^\nu{}_\alpha\, {}^{(p-1)}Y^\alpha. \tag{1.5b}$$

THEOREM 1.1. *The SE-connection $\Gamma^\nu_{\lambda\mu}$ of ${}^*g$-SEX$_n$ may be given by*

$$\Gamma^\nu_{\lambda\mu} = {}^*\{{}^{\,\nu}_{\lambda\mu}\} + S_{\lambda\mu}{}^\nu + U^\nu{}_{\lambda\mu}, \tag{1.6a}$$

where ${}^*\{{}^{\,\nu}_{\lambda\mu}\}$ is the Christoffel symbols defined by ${}^*h_{\lambda\mu}$, and

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda} X_{\mu]}, \quad U^\nu{}_{\lambda\mu} = -{}^*h_{\lambda\mu}\, {}^{(2)}X^\nu. \tag{1.6b}$$

Let

$$S_\lambda \overset{\text{def}}{=} S_{\lambda\alpha}{}^\alpha, \quad U_\lambda \overset{\text{def}}{=} U^\alpha{}_{\lambda\alpha}. \tag{1.7}$$

THEOREM 1.2. *In $^*g$-SEX_n under the present conditions*[1], *the vectors X_λ, S_λ, and U_λ are involved in the following identities, for $p, q = 1, 2, 3, \cdots$:*

$$S_\lambda = (1-n)X_\lambda \tag{1.8a}$$

$$U_\lambda = -{}^{(2)}X_\lambda = -\frac{1}{2}\,\partial_\lambda ln\, {}^*g, \qquad \text{where } {}^*g \stackrel{\text{def}}{=} \frac{det({}^*g_{\lambda\mu})}{det({}^*h_{\lambda\mu})} \tag{1.8b}$$

$${}^{(p+1)}S_\lambda = (1-n)\,{}^{(p+1)}X_\lambda = (n-1)\,{}^{(p)}U_\lambda \tag{1.8c}$$

$${}^{(p)}U_\alpha\,{}^{(q)}X^\alpha = (-1)^{p+1}\,{}^{(p+q-1)*}k_{\beta\gamma}\,X^\beta X^\gamma \tag{1.8d}$$

$${}^{(p)}U_\alpha\,{}^{(q)}X^\alpha = 0, \qquad \text{if } p+q-1 \text{ is odd} \tag{1.8e}$$

$$D_\lambda X_\mu = \nabla_\lambda X_\mu \tag{1.8f}$$

$$D_{[\lambda}X_{\mu]} = \nabla_{[\lambda}X_{\mu]} = \partial_{[\lambda}X_{\mu]} \tag{1.8g}$$

$$\nabla_{[\lambda}U_{\mu]} = 0, \quad D_{[\lambda}U_{\mu]} = 2U_{[\lambda}X_{\mu]} = -2\,{}^{(2)}X_{[\lambda}X_{\mu]} \tag{1.8h}$$

where ∇_λ is the symbolic vector of the covariant derivative with respect to ${}^*\{{}^{\nu}_{\lambda\mu}\}$.

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[1] The situation that the conditions in Agreement I.2.5 are imposed on our $^*g$-SEX$_n$ are described in this paper by the words *"present conditions"*.

THEOREM 1.3. *Under the present conditions, the SE-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $^*g$-SEX$_n$ may be given by*

$$R_{\omega\mu\lambda}{}^{\nu} = {}^*H_{\omega\mu\lambda}{}^{\nu} + M_{\omega\mu\lambda}{}^{\nu} + N_{\omega\mu\lambda}{}^{\nu}, \tag{1.9}$$

where

$${}^*H_{\omega\mu\lambda}{}^{\nu} = 2(\partial_{[\mu}{}^*\{{}_{\omega]\lambda}^{\ \nu}\} + {}^*\{{}_{\alpha[\mu}^{\ \nu}\}{}^*\{{}_{\omega]\lambda}^{\ \alpha}\}), \tag{1.10a}$$

$$M_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{\lambda}\partial_{[\mu}X_{\omega]} + \delta^{\nu}_{[\mu}\nabla_{\omega]}X_{\lambda} - {}^*h_{\lambda[\omega}\nabla_{\mu]}{}^{(2)}X^{\nu}), \tag{1.10b}$$

$$N_{\omega\mu\lambda}{}^{\nu} = 2(\delta^{\nu}_{[\mu}X_{\omega]}X_{\lambda} + {}^*h_{\lambda[\omega}{}^{(2)}X_{\mu]}{}^{(2)}X^{\nu}). \tag{1.10c}$$

THEOREM 1.4 (Generalized Bianchi's identity in $^*g$-SEX$_n$). *Under the present conditions, the SE-curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $^*g$-SEX$_n$ satisfies the following identity:*

$$D_{[\xi}R_{\omega\mu]\lambda}{}^{\nu} = -4X_{[\xi}{}^*H_{\omega\mu]\lambda}{}^{\nu} + Z_{[\xi\omega\mu]\lambda}{}^{\nu}, \tag{1.11a}$$

where

$$\begin{aligned}\frac{1}{8}Z_{\xi\omega\mu\lambda}{}^{\nu} = \{&\delta^{\nu}_{\lambda}X_{\xi}\,\partial_{\omega}X_{\mu} + X_{\xi}\,\delta^{\nu}_{\omega}\,\nabla_{\mu}X_{\lambda} \\ &- X_{\xi}\,\nabla_{\omega}({}^*h_{\mu\lambda}{}^{(2)}X^{\nu})\} - {}^*h_{\lambda\xi}\,X_{\omega}{}^{(2)}X_{\mu}{}^{(2)}X^{\nu}.\end{aligned} \tag{1.11b}$$

2. The contracted SE-curvature tensors

This section is devoted to the study of the contracted n-dimensional SE-curvature tensors, defined by the SE-connection in $^*g$-UFT under the present conditions, and of some useful identities involving them.

The tensors

$$R_{\mu\lambda} \overset{\text{def}}{=} R_{\alpha\mu\lambda}{}^{\alpha}, \quad V_{\omega\mu} \overset{\text{def}}{=} R_{\omega\mu\alpha}{}^{\alpha} \tag{2.1}$$

are called *the first and the second contracted SE-curvature tensors* of the SE-connection $\Gamma^{\nu}_{\lambda\mu}$, respectively. We see in the following two theorems that they appear as functions of the vectors X_{λ}, S_{λ}, U_{λ}, and hence also as functions of $^*g_{\lambda\mu}$ and its first two derivatives in virtue of (1.8a, b) and I (2.25).

THEOREM 2.1. *The second contracted SE-curvature tensor $V_{\omega\mu}$ in $^{*}g$-SEX_n under the present conditions is a curl of the vector S_λ. That is,*

$$V_{\omega\mu} = 2\,\partial_{[\omega} S_{\mu]}. \tag{2.2}$$

Proof. Putting $\lambda = \nu = \alpha$ in (1.9), we have

$$V_{\omega\mu} = {}^{*}H_{\omega\mu\alpha}{}^{\alpha} + M_{\omega\mu\alpha}{}^{\alpha} + N_{\omega\mu\alpha}{}^{\alpha}. \tag{2.3}$$

In virtue of I(3.12) and (1.8a, b, h), the relations (1.10a, b, c) give

$$^{*}H_{\omega\mu\alpha}{}^{\alpha} = N_{\omega\mu\alpha}{}^{\alpha} = 0$$

$$M_{\omega\mu\alpha}{}^{\alpha} = -2n\,\partial_{[\omega} X_{\mu]} + 2\nabla_{[\omega} X_{\mu]} = 2(1-n)\partial_{[\omega} X_{\mu]} = 2\partial_{[\omega} S_{\mu]},$$

which together with (2.3) proves our assertion. □

THEOREM 2.2. *The first contracted SE-curvature tensor $R_{\mu\lambda}$ in $^{*}g$-SEX_n under the present conditions may be given by*

$$\begin{aligned} R_{\mu\lambda} = {}^{*}H_{\mu\lambda} + 2\,\partial_{[\mu} X_{\lambda]} + \nabla_\mu T_\lambda - {}^{*}h_{\mu\lambda}(\nabla_\alpha U^\alpha + U_\alpha U^\alpha) \\ + (1-n) X_\mu X_\lambda + U_\mu U_\lambda, \end{aligned} \tag{2.4}$$

where

$$^{*}H_{\mu\lambda} \overset{\text{def}}{=} {}^{*}H_{\alpha\mu\lambda}{}^{\alpha} \tag{2.5a}$$

$$T_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=} S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}, \quad T_\lambda \overset{\text{def}}{=} T_{\lambda\alpha}{}^{\alpha} = S_\lambda + U_\lambda. \tag{2.5b}$$

Proof. Putting $\omega = \nu = \alpha$ in (1.9) and making use of (2.5a), we have

$$R_{\mu\lambda} = {}^{*}H_{\mu\lambda} + M_{\alpha\mu\lambda}{}^{\alpha} + N_{\alpha\mu\lambda}{}^{\alpha}. \tag{2.6}$$

In virtue of (1.8a, b), it follows from (1.10b) that

(2.7a)
$$\begin{aligned} M_{\alpha\mu\lambda}{}^{\alpha} &= 2\,\partial_{[\mu} X_{\lambda]} + (1-n)\nabla_\mu X_\lambda + {}^{*}h_{\mu\lambda}\,\nabla_\alpha {}^{(2)}X^\alpha - \nabla_\mu {}^{(2)}X_\lambda \\ &= 2\,\partial_{[\mu} X_{\lambda]} + \nabla_\mu T_\lambda - {}^{*}h_{\mu\lambda}\,\nabla_\alpha U^\alpha. \end{aligned}$$

On the other hand, in virtue of (1.8b) the relation (1.10c) gives

$$\begin{aligned} N_{\alpha\mu\lambda}{}^{\alpha} &= (1-n) X_\mu X_\lambda + {}^{(2)}X_\mu {}^{(2)}X_\lambda - {}^{*}h_{\mu\lambda}\,{}^{(2)}X_\alpha {}^{(2)}X^\alpha \\ &= (1-n) X_\mu X_\lambda + U_\mu U_\lambda - {}^{*}h_{\mu\lambda}\,U_\alpha U^\alpha. \end{aligned} \tag{2.7b}$$

Our assertion follows immediately from (2.6) and (2.7a, b). □

THEOREM 2.3. *The tensor $R_{\mu\lambda}$ is symmetric when $n = 3$.*

Proof. The relation (2.4) may be written as

$$R_{\mu\lambda} = {}^*H_{\mu\lambda} + (3-n)\nabla_\mu X_\lambda - 2\nabla_{(\mu}X_{\lambda)} + \nabla_\mu U_\lambda \\ - {}^*h_{\mu\lambda}(\nabla_\alpha U^\alpha + U_\alpha U^\alpha) + (1-n)X_\mu X_\lambda + U_\mu U_\lambda, \tag{2.8}$$

where use has been made of (1.8a, g) and (2.5b). Hence, in virtue of (1.8g, h) we have

$$R_{[\mu\lambda]} = 0 \quad \text{if and only if} \quad (3-n)\nabla_{[\mu}X_{\lambda]} = (3-n)\partial_{[\mu}X_{\lambda]} = 0$$

from which our assertion follows. □

REMARK 2.4. In the proof of the Theorem (2.3), we excluded the case that $\partial_{[\mu}X_{\lambda]} = 0$, because we assumed that X_λ is not a gradient vector in the definition of semi-symmetric connection in (1.4). In fact, the assumption that X_λ is not a gradient vector is essential in the discussions of the field equations in $^*g$-SEX$_n$.

THEOREM 2.5. *The contracted SE-curvature tensors in $^*g$-SEX$_n$ under the present conditions are related by*

$$2R_{[\mu\lambda]} = 4\,\partial_{[\mu}X_{\lambda]} + V_{\mu\lambda}. \tag{2.9}$$

Proof. In virtue of (1.8a, g, h), the relation (2.9) may be proved from (2.8) as in the following way:

$$\begin{aligned} 2R_{[\mu\lambda]} &= 2(3-n)\partial_{[\mu}X_{\lambda]} \\ &= 2(1-n)\partial_{[\mu}X_{\lambda]} + 4\,\partial_{[\mu}X_{\lambda]} \\ &= 2\,\partial_{[\mu}S_{\lambda]} + 4\,\partial_{[\mu}X_{\lambda]} \\ &= V_{\mu\lambda} + 4\,\partial_{[\mu}X_{\lambda]}. \end{aligned}$$

□

REMARK 2.6. An alternative proof of the Theorem (2.5) may be obtained as follows: Putting $\lambda = \nu = \alpha$ in the identity I (4.6)

$$R_{[\omega\mu\lambda]}{}^{\nu} = 4\,\delta^{\nu}_{[\lambda}\partial_{\mu}X_{\omega]}$$

and using (1.8a), we have

$$V_{\omega\mu} - R_{\mu\omega} + R_{\omega\mu} = 4\,\partial_{[\omega}S_{\mu]} + 4\,\partial_{[\omega}X_{\mu]},$$

from which the relation (2.9) follows.

Our next task is to obtain a generalization of the classical identity

$$\nabla_{\alpha}E_{\mu}{}^{\alpha} = 0, \tag{2.10}$$

where

$$^{*}H \overset{\text{def}}{=} {}^{*}h^{\alpha\beta}\ {}^{*}H_{\alpha\beta}, \quad E_{\mu}{}^{\nu} \overset{\text{def}}{=} {}^{*}H_{\mu}{}^{\nu} - \frac{1}{2}\delta^{\nu}_{\mu}\ {}^{*}H. \tag{2.11}$$

REMARK 2.7. The tensor $E_{\mu}{}^{\nu}$ is called the *Einstein tensor.* This tensor has a great deal of applications in physics. It is of fundamental importance since its divergence vanishes identically as we see in (2.10).

In our further considerations, the quantities

$$R \overset{\text{def}}{=} {}^{*}h^{\alpha\beta}R_{\alpha\beta}, \quad G_{\mu}{}^{\nu} \overset{\text{def}}{=} R_{\mu}{}^{\nu} - \frac{1}{2}\delta^{\nu}_{\mu}R \tag{2.12}$$

will be referred to *SE-curvature invariant and SE-Einstein tensor of* $^{*}g$-SEX_n, respectively. The tensor $G_{\mu}{}^{\nu}$ is the generalized concept of $E_{\mu}{}^{\nu}$. First of all, we need the following two theorems in order to generalize the identity (2.10) in $^{*}g$-SEX$_n$ under the present conditions.

THEOREM 2.8. *In* $^{*}g$-SEX_n *under the present conditions, we have*

$$D_{\omega}{}^{*}h^{\lambda\nu} = 2\,{}^{*}h^{\lambda\nu}X_{\omega} - 2\,\delta^{(\lambda}_{\omega}(X^{\nu)} + {}^{(2)}X^{\nu)}). \tag{2.13a}$$

In particular,

$$D_{\alpha}{}^{*}h^{\lambda\alpha} = S^{\lambda} + (n+1)U^{\lambda}. \tag{2.13b}$$

Proof. Substituting (1.4) into (1.3) for $S_{\omega\alpha}{}^{\nu}$ and making use of (1.2b) and (1.5b), the relation (2.13a) follows as in the following way:

$$\begin{aligned} & D_\omega {}^*h^{\lambda\nu} \\ &= D_\omega {}^*g^{(\lambda\nu)} = -2\, S_{\omega\alpha}{}^{(\nu\, *}g^{\lambda)\alpha} \\ &= (\delta^\nu_\alpha X_\omega - \delta^\nu_\omega X_\alpha)({}^*h^{\lambda\alpha} + {}^*k^{\lambda\alpha}) + (\delta^\lambda_\alpha X_\omega - \delta^\lambda_\omega X_\alpha)({}^*h^{\nu\alpha} + {}^*k^{\nu\alpha}) \\ &= 2\, {}^*h^{\lambda\nu} X_\omega - 2\, \delta^{(\lambda}_\omega (X^{\nu)} + {}^{(2)}X^{\nu)}). \end{aligned}$$

In virtue of (1.8a, b), the relation (2.13b) is a direct consequence of (2.13a). □

THEOREM 2.9. *In $^*g$-SEX$_n$ under the present conditions, we have*

(2.14a) $$R = {}^*H + (1-n)(\nabla_\alpha X^\alpha + \nabla_\alpha U^\alpha + U + X),$$

(2.14b) $$D_\alpha R_\mu{}^\alpha = \nabla_\alpha R_\mu{}^\alpha + (U_\alpha - nX_\alpha) R_\mu{}^\alpha + RX_\mu - U^\alpha R_{\alpha\mu},$$

where

(2.14c) $$X \stackrel{\text{def}}{=} X_\alpha X^\alpha, \quad U \stackrel{\text{def}}{=} U_\alpha U^\alpha.$$

Proof. In virtue of (2.11), (2.12), and (1.8a, g), the representation (2.14a) follows from (2.4). On the other hand, the representation (2.14b) may be proved as in the following way in virtue of (1.6a, b), (1.7), (1.8a, b), and (2.12):

$$\begin{aligned} & D_\alpha R_\mu{}^\alpha \\ &= \partial_\alpha R_\mu{}^\alpha + \Gamma^\alpha_{\beta\alpha} R_\mu{}^\beta - \Gamma^\beta_{\mu\alpha} R_\beta{}^\alpha \\ &= \nabla_\alpha R_\mu{}^\alpha + (S_\beta + U_\beta) R_\mu{}^\beta - S_{\mu\alpha}{}^\beta R_\beta{}^\alpha - U^\beta{}_{\mu\alpha} R_\beta{}^\alpha \\ &= \nabla_\alpha R_\mu{}^\alpha + \{(1-n)X_\alpha + U_\alpha\} R_\mu{}^\alpha + 2\delta^\beta_{[\alpha} X_{\mu]} R_\beta{}^\alpha - {}^*h_{\mu\alpha} U^\beta R_\beta{}^\alpha \\ &= \nabla_\alpha R_\mu{}^\alpha + (U_\alpha - nX_\alpha) R_\mu{}^\alpha + RX_\mu - U^\alpha R_{\alpha\mu}. \end{aligned}$$ □

Now we are ready to prove the following generalization of (2.10).

THEOREM 2.10a (First variation of the generalized Bianchi's identity in $^*g$-SEX$_n$). *The SE-Einstein tensor* $G_\mu{}^\nu$ *satisfies the following identity in* $^*g$-*SEX*$_n$ *under the present conditions:*

(2.15a)
$$D_\alpha G_\mu{}^\alpha = P_\mu - \frac{1}{2}\partial_\mu M,$$

where
(2.15b)
$$P_\mu = \nabla_\alpha(R_\mu{}^\alpha - {}^*H_\mu{}^\alpha) + (U_\alpha - nX_\alpha)R_\mu{}^\alpha + RX_\mu - U^\alpha R_{\alpha\mu},$$

(2.15c)
$$M = \frac{1-n}{2}(\nabla_\alpha X^\alpha + \nabla_\alpha U^\alpha + U + X).$$

Proof. The relation (2.12) gives

(2.16)
$$D_\alpha G_\mu{}^\alpha = D_\alpha R_\mu{}^\alpha - \frac{1}{2}\partial_\mu R.$$

The proof of the identity (2.15a) immediately follows by substituting (2.14a, b) into (2.16) and making use of (2.15b, c) and the following classical identity:
$$\nabla_\alpha {}^*H_\mu{}^\alpha = \frac{1}{2}\partial_\mu {}^*H.$$
□

THEOREM 2.10b (Second variation of the generalized Bianchi's identity in $^*g$-SEX$_n$). *The SE-Einstein tensor* $G_\mu{}^\nu$ *satisfies the following identity in* $^*g$-*SEX*$_n$ *under the present conditions:*

(2.17a)
$$\begin{aligned} 2D_\alpha G_\mu{}^\alpha =& X^\alpha C_{\mu\alpha} + U^\alpha D_{\mu\alpha} - 2RX_\mu + \partial_\mu R \\ &- 3\,{}^*h^{\omega\lambda} Z_{[\beta\omega\mu]\lambda}{}^\beta + 2\,{}^*h^{\omega\lambda} D_\beta(R_{\omega\mu(\lambda\alpha)}\,{}^*h^{\alpha\beta}), \end{aligned}$$

where $Z_{\xi\omega\mu\lambda}{}^\nu$ *is given by* (1.11b), *and*

(2.17b)
$$C_{\mu\alpha} = (4-n)R_{\mu\alpha} + R_{\alpha\mu} - V_{\alpha\mu} - 8E_{\mu\alpha} - R_{\beta\mu\gamma\alpha}\,{}^*h^{\gamma\beta},$$

(2.17c)
$$D_{\mu\alpha} = -R_{\alpha\mu} + n\,R_{\mu\alpha} + V_{\alpha\mu} + R_{\beta\mu\gamma\alpha}\,{}^*h^{\gamma\beta}.$$

Proof. The proof of this theorem is based on the generalized Bianchi's identity (1.11). It may be written in the form

$$-D_\xi(R_{\omega\mu\alpha\lambda}\,{}^*h^{\nu\alpha}) + D_\omega\, R_{\mu\xi\lambda}{}^{\nu} + D_\mu\, R_{\xi\omega\lambda}{}^{\nu}$$
$$= -12\, X_{[\xi}\,{}^*H_{\omega\mu]\lambda}{}^{\nu} + 3\, Z_{[\xi\omega\mu]\lambda}{}^{\nu} - 2\, D_\xi(R_{\omega\mu(\lambda\alpha)}\,{}^*h^{\nu\alpha}).$$

If we contract for ν and ξ and multiply ${}^*h^{\omega\lambda}$ to both sides of the above equation, we have
(2.18)
$$-\,{}^*h^{\omega\lambda}\, D_\beta(R_{\omega\mu\alpha\lambda}\,{}^*h^{\alpha\beta}) - {}^*h^{\omega\lambda}\, D_\omega\, R_{\mu\lambda} + {}^*h^{\omega\lambda}\, D_\mu\, R_{\omega\lambda}$$
$$= -12\,{}^*h^{\omega\lambda}\, X_{[\beta}\,{}^*H_{\omega\mu]\lambda}{}^{\beta} + 3\,{}^*h^{\omega\lambda}\, Z_{[\beta\omega\mu]\lambda}{}^{\beta} - 2\,{}^*h^{\omega\lambda}\, D_\beta\,(R_{\omega\mu(\lambda\alpha)}\,{}^*h^{\alpha\beta})$$

In virtue of (1.8b), (2.1), (2.12), and (2.13a), the terms on the left-hand side of (2.18) are the ones to be rewritten as

(2.19a)
the first term
$$= -D_\beta(R_{\omega\mu\alpha\lambda}\,{}^*h^{\alpha\beta}\,{}^*h^{\omega\lambda}) + R_{\omega\mu\alpha\lambda}\,{}^*h^{\alpha\beta}\, D_\beta\,{}^*h^{\omega\lambda}$$
$$= -D_\beta\, R_\mu{}^{\beta} + 2R_{\omega\mu\alpha\lambda}\,{}^*h^{\alpha\beta}{}^*h^{\omega\lambda}\, X_\beta - \delta_\beta^{(\omega}(X^{\lambda)} + {}^{(2)}X^{\lambda)})$$
$$= -D_\alpha\, R_\mu{}^{\alpha} + 2R_{\mu\alpha}\, X^\alpha - V_{\alpha\mu}(X^\alpha - U^\alpha) - R_{\beta\mu\gamma\alpha}\,{}^*h^{\gamma\beta}(X^\alpha - U^\alpha),$$

(2.19b)
the second term
$$= -D_\omega\, R_\mu{}^{\omega} + R_{\mu\lambda}\, D_\omega\,{}^*h^{\omega\lambda}$$
$$= -D_\alpha\, R_\mu{}^{\alpha} + (1-n)R_{\mu\alpha}\, X^\alpha + (n+1)R_{\mu\alpha}\, U^\alpha,$$

(2.19c)
the third term
$$= D_\mu R - R_{\omega\lambda}\, D_\mu\,{}^*h^{\omega\lambda}$$
$$= D_\mu R - R_{\omega\lambda}\,(2{}^*h^{\omega\lambda}\, X_\mu - 2\,\delta_\mu^{(\omega}(X^{\lambda)} + {}^{(2)}X^{\lambda)})$$
$$= D_\mu R - 2RX_\mu + R_{\mu\alpha}\, X^\alpha + R_{\alpha\mu}\, X^\alpha - R_{\mu\alpha}\, U^\alpha - R_{\alpha\mu}\, U^\alpha.$$

On the other hand, the relations (2.5a) and (2.11) allow the first term on the right-hand side of (2.18) to be expressed in the form

$$-12\,{}^*h^{\omega\lambda}\,X_{[\beta}\,{}^*H_{\omega\mu]\lambda}{}^{\beta} = 8X_\alpha\,E_\mu{}^\alpha. \tag{2.19d}$$

We now substitute (2.19a, b, c, d) into (2.18) to complete the proof of (2.17). □

REMARK 2.11. Several earlier authors, such as Bose (1953), Einstein (1955), Lichnerowicz (1955), Schrödinger (1949), and Winorgradski (1956) tried to generalize (2.10) on a manifold X_n to which Einstein's connection is connected, but their results are cumbersome. Note that our result (2.15) in Theorem (2.10), which holds on the manifold ${}^*g$-SEX$_n$ under the present conditions, is very handy and surveyable tensorial form. On the other hand, comparing the expressions (2.15) and (2.17), we note that the former is more refined. The identity (2.17) contains the last two terms which are too complicated.

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