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ON LEFT DERIVATIONS AND DERIVATIONS OF BANACH ALGEBRAS

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ABSTRACT. In this paper we show that every left derivation on a semiprime Banach algebra A is a derivation which maps A into the intersection of the center of A and the Jacobson radical of A, and hence every left derivation on a semisimple Banach algebra is always zero.

1. Introduction

In 1955 Singer and Wermer proved that the range of a continuous derivation on a commutative Banach algebra is contained in the Jacobson radical [9]. In the same paper they conjectured that the assumption of continuity is not necessary. In 1988 Thomas proved the Singer-Wermer conjecture [10]. Hence, derivations on Banach algebras (if everywhere defined) genuinely belongs to the non-commutative setting. The non-commutative version of the Singer-Wermer theorem is related to the commutator relation. There are various non-commutative versions of the Singer-Wermer theorem. For example, in [1] Brešar and Vukman proved that every continuous left derivation on a Banach algebra A maps A into its Jacobson radical. Also they proved that every left derivation on a semiprime ring X is a derivation which maps X into its center. The main purpose of this paper is to show that every left derivation on a semiprime Banach algebra A is a derivation which maps A into the intersection of the center of A and the Jacobson radical of A, and hence every left derivation on a semisimple Banach algebra

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is always zero. Using this main result, we also show some results of derivations and left derivations.

2. Preliminaries

Throughout, A will represent a complex algebra with center $Z(A)$, R the Jacobson radical of A. Recall that A is prime if $xAy = 0$ implies $x = 0$ or $y = 0$, and A is semiprime if $xAx = 0$ implies $x = 0$. A linear mapping $D : A \to A$ is called a derivation if $D(xy) = xD(y) + D(x)y$ $(x, y \in A)$. A linear mapping $D : A \to A$ is called a left derivation if $D(xy) = xD(y) + yD(x)$ $(x, y \in A)$. Let T be a linear mapping from a Banach space X into a Banach space Y. Then the separating space of T is defined as

$$S(T) = \{y \in Y : \text{ there exists } x_k \to 0 \text{ in } X \text{ with } T(x_k) \to y\},$$

and T is continuous if and only if $S(T) = \{0\}$ (see [8]). $\mathbb{N}$ will denote the set of all natural numbers.

3. The results

DEFINITION 3.1. Let A be a Banach algebra. A closed 2-sided ideal J of A is a *separating ideal* if for each sequence $\{a_n\}$ in A, there exists $m \in \mathbb{N}$ such that $\overline{(Ja_n \dots a_1)} = \overline{(Ja_m \dots a_1)}$ for all $n \geq m$.

By Stability Lemma [3] it is easy to see that every derivation on a Banach algebra has a separating space which is a separating ideal.

The following lemma is due to Cusack [2].

LEMMA 3.2. *Let A be a Banach algebra, and P a minimal prime ideal of A such that $J \not\subset P$, where J is a separating ideal of A. Then P is closed.*

The following lemma can be referred to [5].

LEMMA 3.3. *Let D be a left derivation on an algebra A. Then*

$$D^n(xy) = \sum_{r=0}^{n-1} \binom{n-1}{r} [D^r(x)D^{n-r}(y) + D^r(y)D^{n-r}(x)] (n \in \mathbb{N})$$

holds for all $x, y \in A$.

The following lemma is a crucial tool in proving Lemma 3.5.

LEMMA 3.4. *Let D be a left derivation on an algebra A. Suppose that P is a minimal prime ideal of A such that $[D^r(x), y] \in P$ for all $x, y \in A$ and $r \in \mathbb{N}$, where $[u, v]$ denotes the commutator $uv - vu$. Then $D(P) \subset P$.*

Proof. We shall prove that the ideal $P' = \{a \in P : D^k(a) \in P \text{ for all } k \in \mathbb{N}\}$ is prime again. Since $D(P') \subset P'$, minimality of P therefore yields $D(P) \subset P$. Let $P' \neq \{0\}$. Take $a, b \in A$ such that $a \notin P'$ but $axb \in P'$ for all $x \in A$. Choose $n \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\})$ with the property $D^n(a) \notin P$ and $D^m(a) \in P$ for all $m \in \mathbb{N}_0$, $m < n$. We have to prove by induction that $D^k(b) \in P$ for all $k \in \mathbb{N}_0$. Using Lemma 3.3, we have

$$\begin{aligned}
&D^{n+k}(axb)\\
&= \sum_{j=0}^{n+k-1} \binom{n+k-1}{j} [D^j(a)D^{n+k-j}(xb) + D^j(xb)D^{n+k-j}(a)]\\
&= \sum_{j=0}^{n+k-1} \binom{n+k-1}{j} D^j(a)D^{n+k-j}(xb)\\
&\quad + \sum_{j=0}^{n+k-1} \binom{n+k-1}{j} D^j(xb)D^{n+k-j}(a)
\end{aligned}$$

(1)

$$= \sum_{j=0}^{n-1} \binom{n+k-1}{j} D^j(a)D^{n+k-j}(xb)$$

(2) $$+ \binom{n+k-1}{n} D^n(a)D^k(xb)$$

(3) $$+ \sum_{j=n+1}^{n+k-1} \binom{n+k-1}{j} D^j(a)D^{n+k-j}(xb)$$

$$(4) \qquad + \sum_{j=0}^{k-1} \binom{n+k-1}{j} D^j(xb) D^{n+k-j}(a)$$

$$(5) \qquad + \binom{n+k-1}{k} D^k(xb) D^n(a)$$

$$(6) \qquad + \sum_{j=k+1}^{n+k-1} \binom{n+k-1}{j} D^j(xb) D^{n+k-j}(a)$$

By assumption, the left-hand side always belongs to P. Assume that $k = 0$. Since (1), (6) lie in P and (2), (3), (4) disappear, it follows that $D^n(a)xb \in P$ for all $x \in A$ by the hypothesis of the lemma $[D^r(x), y] \in P$ for all $r \in \mathbb{N}$, which implies that $b \in P$. Now suppose that $k \geq 1$. Then (1) belongs to P since $D^j(a) \in P$ for all $j \leq n-1$. An application of Lemma 3.3 to (3) yields

$$\sum_{j=n+1}^{n+k-1} \binom{n+k-1}{j} D^j(a) D^{n+k-j}(xb)$$

$$= \sum_{j=n+1}^{n+k-1} \binom{n+k-1}{j} D^j(a) \cdot$$

$$\left[\sum_{i=0}^{n+k-j-1} \binom{n+k-j-1}{i} (D^i(x) D^{n+k-j-i}(b) + D^i(b) D^{n+k-j-i}(x)) \right],$$

which belongs to P since $D^i(b) \in P$ and $D^{n+k-j-i}(b) \in P$ for $0 \leq i \leq n+k-j \leq k-1$ by the induction hypothesis. Also another application of Lemma 3.3 to (4) yields

$$\sum_{j=0}^{k-1} \binom{n+k-1}{j} D^j(xb) D^{n+k-j}(a)$$

$$= \sum_{j=0}^{k-1} \binom{n+k-1}{j} \cdot$$

$$\left[\sum_{i=0}^{j-1} \binom{j-1}{i} (D^i(x) D^{j-i}(b) + D^i(b) D^{j-i}(x)) \right] D^{n+k-j}(a),$$

which belongs to P since $D^i(b) \in P$ and $D^{j-i}(b) \in P$ for $0 \leq i \leq j \leq k-1$ by the induction hypothesis. Finally, (6) belongs to P since $D^{n+k-j}(a) \in P$ for $n+k-j \leq n-1$. Hence we have

$$\binom{n+k-1}{n} D^n(a)D^k(xb) + \binom{n+k-1}{k} D^k(xb)D^n(a) \in P.$$

The assumption of the lemma $[D^r(x), y] \in P$ for all $x, y \in A$ and $r \in \mathbb{N}$ gives us

$$\left[\binom{n+k-1}{n} + \binom{n+k-1}{k}\right] D^n(a)D^k(xb) \in P.$$

Thus we obtain $D^n(a)D^k(xb) \in P$. But

$$\begin{aligned} D^n(a)D^k(xb) &= D^n(a)[xD^k(b) + bD^k(x) \\ &\quad + \sum_{i=1}^{k-1} \binom{k-1}{i} (D^i(x)D^{k-i}(b) + D^i(b)D^{k-i}(x))]. \end{aligned}$$

By the induction hypothesis we have $D^i(b) \in P$ and $D^{k-i}(b) \in P$. Consequently, we see that $D^n(a)xD^k(b) \in P$ for all $x \in A$. Since P is a prime ideal, it follows that $D^k(b) \in P$. In case $P' = \{0\}$, we take $a, b \in A$ such that $a \neq 0$ but $axb = 0$ for all $x \in A$. The remainder follows the same fashion as in case $P' \neq \{0\}$. Then we obtain $D^k(b) \in P$ for all $k \in \mathbb{N}_0$, and hence $b = 0$. We complete the proof. □

LEMMA 3.5. *Let D be a left derivation on a Banach algebra A with radical R. Suppose that the following conditions are satisfied:*

(1) *$[D^n(x), y] \in L$ for all $x, y \in A$ and $n \in \mathbb{N}$;*
(2) *$S(D) \subset Z(A)$,*

where $S(D)$ is the separating space of the left derivation D and L is the prime radical of A. Then $D(A) \subset R$.

Proof. Let Q be any primitive ideal of A. Using Zorn's lemma, we find a minimal prime ideal P contained in Q, and hence $D(P) \subset P$ by condition (1) and Lemma 3.4. Suppose first that P is closed. Then we can define a left derivation $\bar{D} : A/P \to A/P$ by $\bar{D}(x+P) = D(x) + P(x \in A)$. Since A/P is prime, Brešar and Vukman's theorem [1] implies that $\bar{D} = 0$ or A/P is commutative. In the second case, $\bar{D}(A/P)$ is contained in the Jacobson radical of A/P by [9] whence, in both cases, $\bar{D}(A/P) \subset Q/P$. Consequently we see that $D(A) \subset Q$. Observe that $S(D)$ is a separating ideal of A by condition (2). If P is not closed, then we see that $S(D) \subset P$ by Lemma 3.2. Denoting $\pi : A \to A/\bar{P}$ the canonical epimorphism, we have, by [8, Chap. 1], $S(\pi \circ D) = \overline{\pi(S(D))} = \{0\}$ whence $\pi \circ D$ is continuous. As a result, $(\pi \circ D)(\bar{P}) = \{0\}$, that is, $D(\bar{P}) \subset \bar{P}$. Hence we can also define a continuous left derivation $\widetilde{D} : A/\bar{P} \to A/\bar{P}$ by $\widetilde{D}(x + \bar{P}) = D(x) + \bar{P}$ $(x \in A)$. Then we see that $\widetilde{D}(A/\bar{P})$ is contained in the Jacobson radical of $A/\bar{P}$ by [1, Theorem 2.1], and hence $\widetilde{D}(A/\bar{P}) \subset Q/\bar{P}$. So we obtain that $D(A) \subset Q$. It follows that $D(A) \subset Q$ for every primitive ideal Q, that is, $D(A) \subset R$. □

Now we prove our main result.

THEOREM 3.6. *Let D be a left derivation on a semiprime Banach algebra A with radical R. Then D is a derivation such that $D(A) \subset Z(A) \cap R$.*

Proof. Note that D is a derivation such that $D(A) \subset Z(A)$ [1, Proposition 1.6]. Since $D(Z(A)) \subset Z(A)$, we obtain $D^n(A) \subset Z(A)$ for all $n \in \mathbb{N}$. Also we see that $S(D) \subset Z(A)$ since $Z(A)$ is a closed subalgebra of A, Therefore, by Lemma 3.5, we have $D(A) \subset R$. Consequently it follows that $D(A) \subset Z(A) \cap R$. □

COROLLARY 3.7. *Let D be a left derivation on a semisimple Banach algebra. Then $D = 0$.*

Using Corollary 3.7, we can obtain the following results of derivations and a Jordan derivation.

COROLLARY 3.8. ([11, Theorem 3.1]) *Let D be a continuous derivation on a Banach algebra A with radical R. If $[D(x), y] \in R$ for all $x, y \in A$, then $D(A) \subset R$.*

Proof. Since a continuous derivation leaves the Jacobson radical invariant, we may assume that A is semisimple and $[D(x), y] = 0$ for all $x, y \in A$. Thus Corollary 3.7 implies that $D = 0$. □

COROLLARY 3.9. ([1, Theorem 2.2.]) *Let D be a continuous Jordan derivation on a Banach algebra A with radical R. If $[D(x), x] \in R$ for all $x \in A$, then $D(A) \subset R$.*

Proof. By [7, Lemma 3.2], $D(R) \subset R$ wherefore we may assume that A is semisimple and $[D(x), x] = 0$ for all $x \in A$. Note that every continuous Jordan derivation on a semisimple Banach algebra is a derivation [7, Theorem 3.3]. Since $[D(x), x] = 0$ for all $x \in A$ is equivalent to $[D(x), y] = 0$ for all $x, y \in A$ by [6, Proposition 2], we see that D is a left derivation on a semisimple Banach algebra A. Hence Corollary 3.7 implies that $D = 0$. □

DEFINITION 3.10. Let A and B be Banach algebras. A linear mapping $T : A \to B$ is called *spectrally bounded* if there is $M > 0$ such that $r(T(x)) \leq Mr(x)$ for all $x \in A$. If $r(T(x)) = r(x)$ for all $x \in A$, we say that T is a *spectral isometry*. If $r(x) = 0$, then x is called *quasinilpotent*. (Herein, $r(x) = \lim_{n\to\infty} ||x^n||^{\frac{1}{n}}$ denotes the spectral radius of the element x.)

Observe that the canonical epimorphism $\pi : A \to A/R$ is a spectral isometry.

The next theorem is a generalization of Brešar and Vukman's theorem [1, Theorem 2.1].

THEOREM 3.11. *Let D be a left derivation on a Banach algebra A with radical R. If D^n is continuous for some $n \in \mathbb{N}$, then $D(A) \subset R$.*

Proof. Note that the quotient algebra A/R is semisimple. Let $x \in A$ and $y \in R$ and observe that $xD(y) = D(xy) - yD(x) \in D(R) + R$. This shows that $D(R) + R$ is a left ideal of A, hence $\pi(D(R))$ is a left ideal of A/R. A simple modification of the proof of Lemma 2.1 in [4] shows that

$\pi(D^m(x^m)) = \pi(m!(D(x))^m)$ holds for all $x \in R$ and $m \in \mathbb{N}$. Since D^n is continuous for some $n \in \mathbb{N}$, we have, for each $x \in R$ and $k \in \mathbb{N}$,

$$\begin{aligned} \|(\pi(D(x)))^{nk}\|^{\frac{1}{nk}} &\leq ((nk)!)^{-\frac{1}{nk}} \|\pi(D^{nk}(x^{nk}))\|^{\frac{1}{nk}} \\ &\leq ((nk)!)^{-\frac{1}{nk}} \|D^n\|^{\frac{1}{n}} \|x\| \to 0 \text{ as } k \to \infty. \end{aligned}$$

This shows that $\pi(D(R))$ is a quasinilpotent left ideal of A/R, therefore, it is contained in the Jacobson radical of A/R. Semisimplicity forces $D(R) \subset R$. Thus we may assume that A is semisimple. Then it follows from Corollary 3.7 that $D = 0$. □

We now have the final result of this paper.

THEOREM 3.12. *Let D be a left derivation on a Banach algebra A with radical R. Then $D(A) \subset R$ if and only if D is spectrally bounded.*

Proof. One way implication is obvious, so suppose that $r(D(x)) \leq Mr(x)$ for some $M > 0$ and all $x \in A$. Then we know that

$$\begin{aligned} r(xD(y)) &= r(D(xy) - yD(x)) \\ &= r(\pi(D(xy) - yD(x))) \\ &= r(\pi(D(xy)) - \pi(yD(x))) \\ &= r(\pi(D(xy))) \\ &= r(D(xy)) \leq Mr(xy) = 0 \end{aligned}$$

whenever $y \in R$ and $x \in A$, whence $D(R) \subset R$. Hence we may assume that A is semisimple. Now, $D = 0$ follows directly from Corollary 3.7.□

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