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CLIFFORD L^2-COHOMOLOGY ON THE COMPLETE KÄHLER MANIFOLDS II

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ABSTRACT. In this paper, we prove that on the complete Kähler manifold, if $\rho(x) \geq -\frac{1}{2}\lambda_0$ and either $\rho(x_0) > -\frac{1}{2}\lambda_0$ at some point x_0 or $Vol(M) = \infty$, then the Clifford L^2-cohomology group $L^2\mathcal{H}^*(M,S)$ is trivial, where $\rho(x)$ is the least eigenvalue of $\mathcal{R}_x + \bar{\mathcal{R}}(x)$ and λ_0 is the infimum of the spectrum of the Laplacian acting on L^2- functions on M.

0. One of the important object in the study of a manifold is its Clifford algebra $Cl(M)$, generated by the tangent space. It carries an intrinsic first order elliptic operator D, which is called the Dirac operator. There is a canonical vector (but not algebra) bundle isomorphism $\Lambda^*(M) \to Cl(M)$, where $\Lambda^*(M)$ is an exterior algebra of M. In $\Lambda^*(M)$, the Dirac operator D is $D \cong d+\delta$ and the Laplace operator is the square of the Dirac operator, where d is the exterior differential and δ is the adjoint operator of d. Therefore many results of the Clifford theory yield the results of the de Rham theory ([8]). In 1980, M. L. Michelsohn ([10]) proved many results for the Dirac operator on compact Kähler manifold. Recently, J. S. Pak and S. D. Jung ([11]) extended the results of M. L. Michelsohn ([10]) and obtained the following theorem for the Dirac operator on complete Kähler manifold.

THEOREM A. *Let M be a complete Kähler manifold and S be any hermitian vector bundle of modules over $\mathbb{C}l(M)$. If R is non-negative and positive at some point of M, then the Clifford L^2-cohomology group*

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is trivial, where R is the symmetric endomorphism of S containing the curvature data.

In this paper, we prove Theorem A under the assumption of weaker curvature endomorphism R which is bounded by $-\frac{1}{2}\lambda_0$ from below, λ_0 is the infimum of the spectrum of the positive Laplacian Δ^M acting on L^2-functions on M. The method of this study is based on that of P. Bérard ([2]). From our results, we deduce the vanishing theorem for the harmonic forms which extend the results of K. D. Elworthy and S. Rosenberg ([4]) to the Kähler case. Also, we study the harmonic spinors under some condition of the scalar curvature.

1. Let M be a $2n$-dimensional Kähler manifold with almost complex structure J and with connection ∇. Let $Cl(M)$ be the Clifford bundle generated by the tangent bundle TM. Now we define a derivation $\mathcal{J}_0 : Cl(M) \to Cl(M)$ induced by J as follows:

$$\mathcal{J}_0(v_1 \cdots v_k) = \sum_{j=1}^{k} v_1 \cdots Jv_j \cdots v_k \tag{1.1}$$

for $v_1, \cdots, v_k \in TM$, where " $\cdot$ " is the Clifford multiplication. If it is clear from the context which multiplication is meant, we omit the Clifford multiplication " $\cdot$ ". To study $\mathcal{J}_0$ effectively we consider the complexification $\mathbb{C}l(M) = Cl(M) \otimes_{\mathbb{R}} \mathbb{C}$. This algebra has a natural basis given as follows: Let $e_1, \cdots, e_n, Je_1, \cdots, Je_n$ be an orthonormal basis of T_xM. Let $T_x^{1,0}$ (resp. $T_x^{0,1}$) be the i eigenspace (resp. $-i$ eigenspace) of J in $T_xM \otimes \mathbb{C}$. Put

$$\xi_k = \frac{1}{2}\{e_k - iJe_k\}, \quad \bar{\xi}_k = \frac{1}{2}\{e_k + iJe_k\}.$$

Then $\xi_1, \cdots, \xi_n$ (resp. $\bar{\xi}_1, \cdots, \bar{\xi}_n$) is the basis of $T_x^{1,0}$ (resp. $T_x^{0,1}$). And $\{\xi_k, \bar{\xi}_k\}$ has the following properties;

$$\xi_k\bar{\xi}_\ell + \bar{\xi}_k\xi_\ell = \xi_k\bar{\xi}_\ell + \bar{\xi}_\ell\xi_k = -\delta_{k\ell}, \quad \xi_k\xi_\ell = -\xi_\ell\xi_k, \quad \bar{\xi}_k\bar{\xi}_\ell = -\bar{\xi}_\ell\bar{\xi}_k. \tag{1.2}$$

Denote $\xi_K\bar{\xi}_I = \xi_{k_1}\cdots\xi_{k_r}\bar{\xi}_{i_1}\cdots\bar{\xi}_{i_s}$, where K and I range over all strictly ascending multiindices from $\{1,\cdots,n\}$. For convenience we set $\mathcal{J} = \frac{1}{i}\mathcal{J}_0$. Then by the derivation property, we have

(1.3) $$\mathcal{J}(\xi_K\bar{\xi}_I) = (|K| - |I|)\xi_K\bar{\xi}_I,$$

where $|K|, |I|$ denote the lengths of K and I. This gives a decomposition

$$\mathbb{C}l(M) = \bigoplus_{p=-n}^{n} \mathbb{C}l^p(M),$$

where $\mathbb{C}l^p(M) = \{\phi \in \mathbb{C}l(M) \mid \mathcal{J}\phi = p\phi\}$.

We now introduce two intrinsically defined linear maps $\mathcal{L}, \bar{\mathcal{L}} : \mathbb{C}l(M) \to \mathbb{C}l(M)$ as follows; For any $\varphi \in \mathbb{C}l(M)$, set

(1.4) $$\mathcal{L}(\varphi) = -\sum_{k=1}^{n} \xi_k\varphi\bar{\xi}_k, \quad \bar{\mathcal{L}}(\varphi) = -\sum_{k=1}^{n} \bar{\xi}_k\varphi\xi_k.$$

These operators are independent of the Hermitian basis chosen to define them. We consider the operator $\mathcal{H} = [\mathcal{L}, \bar{\mathcal{L}}]$. Then they satisfy the following relations;

(1.5) $$[\mathcal{L}, \bar{\mathcal{L}}] = \mathcal{H}, \quad [\mathcal{H}, \mathcal{L}] = 2\mathcal{L}, \quad [\mathcal{H}, \bar{\mathcal{L}}] = -2\bar{\mathcal{L}}.$$

Hence they define a representation of $s\ell(2,\mathbb{C})$, the Lie algebra of $SL(2,\mathbb{C})$ on $\mathbb{C}l(M)$. Since each of the operators $\mathcal{L}, \bar{\mathcal{L}}$ and $\mathcal{H}$ commutes with $\mathcal{J}$, we can define the subspaces

$$\mathbb{C}l^{p,q}(M) = \{\varphi \in \mathbb{C}l(M) \mid \mathcal{H}\varphi = q\varphi,\ \mathcal{J} = p\varphi\}$$

and obtain a decomposition ([10])

(1.6) $$\mathbb{C}l(M) = \bigoplus_{p,q} \mathbb{C}l^{p,q}(M).$$

PROPOSITION 1.1 ([10]). *For each $\xi \in T^{1,0}(M)$, one has that $\xi \cdot \mathbb{C}l^{p,q} \subseteq \mathbb{C}l^{p+1,q+1}$ and $\bar{\xi} \cdot \mathbb{C}l^{p,q} \subseteq \mathbb{C}l^{p-1,q-1}$. Furthermore, if $\xi \neq 0$, the sequences*

$$\cdots \xrightarrow{\lambda_\xi} \mathbb{C}l^{p-1,q-1} \xrightarrow{\lambda_\xi} \mathbb{C}l^{p,q} \xrightarrow{\lambda_\xi} \mathbb{C}l^{p+1,q+1} \longrightarrow \cdots$$
$$\cdots \xleftarrow{\lambda_{\bar\xi}} \mathbb{C}l^{p-1,q-1} \xleftarrow{\lambda_{\bar\xi}} \mathbb{C}l^{p,q} \xleftarrow{\lambda_{\bar\xi}} \mathbb{C}l^{p+1,q+1} \longleftarrow \cdots,$$

where λ_ξ denotes left Clifford multiplication by ξ, are exact.

2. Suppose that M is a complete Kähler manifold. We introduce two differential operators $\mathcal{D}, \bar{\mathcal{D}} : \Gamma\mathbb{C}l(M) \to \Gamma\mathbb{C}l(M)$ by the formulas

$$\mathcal{D} = \sum_j \xi_j \nabla_{\bar{\xi}_j}, \quad \bar{\mathcal{D}} = \sum_j \bar{\xi}_j \nabla_{\xi_j}, \tag{2.1}$$

where ∇ is the canonical connection. Since ∇ preserves the subbundles $\Gamma\mathbb{C}l^{p,q}(M)$, we have

$$\mathcal{D}(\Gamma\mathbb{C}l^{p,q}) \subset \Gamma\mathbb{C}l^{p+1,q+1}, \quad \bar{\mathcal{D}}(\Gamma\mathbb{C}l^{p,q}) \subset \Gamma\mathbb{C}l^{p-1,q-1}$$

for all p and q. Then we have the following well known fact:

THEOREM 2.1 ([10]). *The operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are formal adjoints of one another on $\Gamma_{cpt}\mathbb{C}l(M)$, the set of all sections with the compact support. And they satisfy*

$$\mathcal{D}^2 = \bar{\mathcal{D}}^2 = 0.$$

Furthermore, the complex

$$\cdots \xrightarrow{\mathcal{D}} \Gamma\mathbb{C}^{p-1,q-1} \xrightarrow{\mathcal{D}} \Gamma\mathbb{C}^{p,q} \xrightarrow{\mathcal{D}} \Gamma\mathbb{C}^{p+1,q+1} \xrightarrow{\mathcal{D}} \cdots$$

is elliptic.

Now we set

$$\Delta := \mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}. \tag{2.2}$$

Then Δ is a formally self-adjoint elliptic operator. To understand Δ we introduce two "real" operators on $\mathbb{C}l(M)$:
(2.3)
$$D = \sum_j \{e_j \nabla_{e_j} + (Je_j)\nabla_{Je_j}\}, \quad D^c = \sum_j \{e_j \nabla_{Je_j} - (Je_j)\nabla_{e_j}\}.$$

The first operator is called the *Dirac operator*. Then we can easily see that

$$\mathcal{D} = \frac{1}{4}(D + iD^c), \quad \bar{\mathcal{D}} = \frac{1}{4}(D - iD^c). \tag{2.4}$$

Since $\mathcal{D}^2 = 0$, we have that $D^2 = (D^c)^2$ and $DD^c + D^cD = 0$. It follows that

$$\Delta = \frac{1}{4}D^2. \tag{2.5}$$

Since D is essentially self-adjoint, we have

$$KerD = KerD^2 = Ker\Delta. \tag{2.6}$$

Now, we consider the usual inner product

$$\langle\langle \varphi_1, \varphi_2 \rangle\rangle = \int_M \langle \varphi_1, \varphi_2 \rangle \tag{2.7}$$

for any $\varphi_1, \varphi_2 \in \Gamma_{cpt}\mathbb{C}l(M)$. Let $L^2(\mathbb{C}l^{p,q}(M))$ be the completion of $\Gamma_{cpt}\mathbb{C}l^{p,q}$ with respect to $\langle\langle \ , \ \rangle\rangle$. We recall that the operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are formal adjoint to one another with respect to $\langle\langle \ , \ \rangle\rangle$. Then $\mathcal{D}$ and $\bar{\mathcal{D}}$ have closed extensions in $L^2(\mathbb{C}l^{p,q}(M))$. But since M is complete, their closed extensions are unique ([3]). From now on, we write the closed extensions as the same symbols. Now, we put

$$L^2\mathcal{H}^{p,q} := Ker\mathcal{D}/\overline{Im\mathcal{D}} \ \cap L^2(\mathbb{C}l^{p,q}(M)), \tag{2.8}$$

$$L^2\hat{\mathcal{H}}^{p,q} := Ker\mathcal{D} \cap Ker\bar{\mathcal{D}} \ \cap L^2(\mathbb{C}l^{p,q}(M)), \tag{2.9}$$

$$L^2H^{p,q} := Ker\Delta \ \cap L^2(\mathbb{C}l^{p,q}(M)). \tag{2.10}$$

Here $L^2\mathcal{H}^{p,q}$ and $L^2H^{p,q}$ are called the *Clifford L^2-cohomology group* and *L^2-harmonic space*, respectively. Then we have

PROPOSITION 2.2 ([11]). *Let M be a complete Kähler manifold. Then we have*

$$L^2\mathcal{H}^{p,q} \cong L^2\hat{\mathcal{H}}^{p,q} \cong L^2H^{p,q}.$$

3. Let M be a Kähler manifold and $S \to M$ a hermitian vector bundle of left modules over $\mathbb{C}l(M)$ with a hermitian metric $\langle\ ,\ \rangle$ such that:

(1) Module multiplication by unit tangent vectors is unitary, i.e.,

$$\langle \xi \cdot \phi, \psi\rangle + \langle \phi, \bar{\xi} \cdot \psi\rangle = 0, \tag{3.1}$$

for any $\phi, \psi \in \Gamma(S)$ and $\xi \in \Gamma(TM) \otimes \mathbb{C}$

(2) With respect to the canonical hermitian connection, covariant differentiation is a derivation of module multiplication. That is, for $\phi \in \Gamma(\mathbb{C}l(M))$ and $s \in \Gamma(S)$, we have

$$\nabla(\phi \cdot s) = (\nabla\phi) \cdot s + \phi \cdot (\nabla s). \tag{3.2}$$

Now, we recall some basic results from [10]. For each j, we set $\omega_j = -\xi_j\bar{\xi}_j$, $\bar{\omega}_j = -\bar{\xi}_j\xi_j$. To each (possibly empty) subset $I = \{i_1, \cdots, i_p\} \subseteq \{1, \cdots, n\}$ with complementary subset $\{j_1, \cdots, j_{n-p}\}$ we set $\omega_I = \omega_{i_1} \cdots \omega_{i_p}\bar{\omega}_{j_1} \cdots \bar{\omega}_{j_{n-p}}$ and we denote $|I| = p$. Then we have

$$1 = \prod_{j=1}^{n}(\omega_j + \bar{\omega}_j) = \sum_{r=1}^{n}\pi_r, \tag{3.3}$$

where $\pi_r = \sum_{|I|=r}\omega_I$. Moreover, we have an orthogonal decomposition of the bundle

$$S = \bigoplus_{r=0}^{n} S^r, \quad S^r = \pi_r \cdot S. \tag{3.4}$$

Then the complex

$$0 \to \Gamma_{cpt}(S^0) \xrightarrow{\mathcal{D}} \Gamma_{cpt}(S^1) \xrightarrow{\mathcal{D}} \cdots \xrightarrow{\mathcal{D}} \Gamma_{cpt}(S^n) \to 0 \tag{3.5}$$

is elliptic and its completion becomes a Hilbert complex ([3]). Similarly with Proposition 2.2, we have

$$L^2\mathcal{H}^r(M,S) \cong L^2\hat{\mathcal{H}}^r(M,S) \cong L^2H^r(M,S). \tag{3.6}$$

Now, we define invariant operators on $\Gamma(S)$ by

$$\begin{aligned} \nabla^*\nabla &= -\sum_j \nabla_{\xi_j,\bar{\xi}_j}, \quad \bar{\nabla}^*\bar{\nabla} = -\sum_j \nabla_{\bar{\xi}_j,\xi_j}, \\ \mathcal{R} &= \sum_{j,k} \xi_j\bar{\xi}_k R_{\bar{\xi}_j,\xi_k}, \quad \bar{\mathcal{R}} = \sum_{j,k} \bar{\xi}_j\xi_k R_{\xi_j,\bar{\xi}_k}, \end{aligned} \tag{3.7}$$

where $R_{V,W} = \nabla_{V,W} - \nabla_{W,V}$ is the curvature tensor and where $\nabla_{V,W} = \nabla_V\nabla_W - \nabla_{\nabla_V W}$ is the invariant second covariant derivative. Then we obtain

PROPOSITION 3.1 ([10]). *For any two sections $s_1,\ s_2 \in \Gamma(S)$, at least one of which has compact support, the following holds:*

$$\int_M \langle \nabla^*\nabla s_1, s_2\rangle = \int_M \langle \nabla s_1, \nabla s_2\rangle,$$

where $\langle \nabla s_1, \nabla s_2\rangle = \langle \nabla_{\bar{\xi}_i} s_1, \nabla_{\bar{\xi}_i} s_2\rangle$. Hence $\nabla^*\nabla$ is a formally self adjoint, nonnegative operator. Similarly, this holds for $\bar{\nabla}^*\bar{\nabla}$. Moreover, the zero order operators $\mathcal{R}$ and $\bar{\mathcal{R}}$ are self-adjoint.

By the straight calculation, we obtain the Bochner-Weitzenböck type formula ([10]);

$$\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D} = \nabla^*\nabla + \mathcal{R} = \bar{\nabla}^*\bar{\nabla} + \bar{\mathcal{R}}. \tag{3.8}$$

From this formula, we have

$$2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}) = \nabla^*\nabla + \bar{\nabla}^*\bar{\nabla} + \mathcal{R} + \bar{\mathcal{R}}. \tag{3.9}$$

Let $\rho(x)$ denote the least eigenvalue of $R_x (= \mathcal{R}_x + \bar{\mathcal{R}}_x)$, the symmetric endomorphism of S_x, that is,

$$\rho(x) = \inf\{\langle R_x(s), s\rangle_{S_x} \mid s \in S_x,\ |s| = 1\}$$

and λ_0 is the infimum of the spectrum of the positive Laplacian Δ^M acting on L^2-functions on M, that is, $\Delta^M = \delta d$, where δ is the adjoint operator of d. Then we have

THEOREM 3.3. *Let M be a complete Kähler manifold and let S be any hermitian vector bundle of modules over $\mathbb{C}l(M)$. If $\rho(x) \geq -\frac{1}{2}\lambda_0$ for all $x \in M$ and either $\rho(x_0) > -\frac{1}{2}\lambda_0$ for some $x_0 \in M$ or (M, g) has infinite volume, then the Clifford L^2-cohomology group is trivial. That is,*

$$L^2\mathcal{H}^r(M, S) = \{0\}, \quad \text{for any } r = 0, 1, \cdots, n.$$

In order to prove that Theorem 3.3, we prepare some Lemmas;

LEMMA 3.4 ([2]) (the first Kato inequality). *For any $s \in \Gamma(S)$, $|d|s|| \leq |\nabla s|$, with equality if and only if for any $X \in TM$, there exists a function f_X such that $\nabla_X s = f_X s$ (at least on the set $\{|s| \neq 0\}$).*

LEMMA 3.5 ([2]). *If $s \in \Gamma(S)$ satisfies $|d|s|| = |\nabla s|$, then on $\{s \neq 0\}$, $s = |s|s_1$, with $\nabla s_1 = 0$.*

LEMMA 3.6 ([2]) (the second Kato inequality). *If $s \in \Gamma(S)$ satisfies $\Delta s = 0$, then $\Delta^M|s| \leq -2\rho|s|$ with equality if and only if $|d|s|| = |\nabla s|$ and $\langle R(s), s\rangle = 2\rho|s|^2$, where $R = \mathcal{R} + \bar{\mathcal{R}}$.*

Proof of Theorem 3.3. By (3.6), it is sufficient to prove that $L^2H^r(M, S) = \{s \in Ker\Delta | s \in L^2(M, S^r)\} = \{0\}$. This proof is based on the method of P. Bérard ([2]). Let $s \in Ker\Delta$ of finite L^2-norm and denote $\phi := |s|$, its pointwise norm. First, we assume that $\rho(x) \geq -\frac{1}{2}\lambda_0$ for all $x \in M$. Using Lemma 3.6, we have

$$\Delta^M\phi \leq -2\rho\phi \leq \lambda_0\phi. \tag{3.10}$$

Since M is complete, one can construct function ω_ℓ such that $\omega_\ell \in C_0^\infty(M)$ and $\omega_\ell \equiv 1$ on $B(x_0, \ell)$, supp $\omega_\ell \subset B(x_0, 2\ell)$ and $|d\omega_\ell| \leq C/\ell$ for some constant C, where $\ell \in \mathbb{R}_+$, $x_0 \in M$ and $B(x_0, \ell)$ is the Riemannian open ball with radius ℓ and center x_0. Multiplying (3.10) by $\omega_\ell^2\phi$ and integrating by parts, we obtain

$$\int \langle d\phi, d\omega_\ell^2\phi\rangle \leq -2\int \rho\omega_\ell^2\phi^2 \leq \lambda_0 \int \omega_\ell^2\phi^2, \tag{3.11}$$

where $\langle \ , \ \rangle$ denotes the hermitian metric on T^*M. By straight calculation, we have the equality

$$\int \omega_\ell^2|d\phi|^2 + 2\int \omega_\ell\phi\langle d\omega_\ell, d\phi\rangle = \int |d(\omega_\ell\phi)|^2 - \int \phi^2|d\omega_\ell|^2. \tag{3.12}$$

Summing (3.11) and (3.12), we obtain
(3.13)
$$\int |d(\omega_\ell\phi)|^2 \le -2\int \rho\omega_\ell^2\phi^2 + \int \phi^2|d\omega_\ell|^2 \le \lambda_0 \int \omega_\ell^2\phi^2 + \int \phi^2|d\omega_\ell|^2.$$

On the other hand, since λ_0 is the infimum of the spectrum of Δ^M, we get

$$\int |d(\omega_\ell\phi)|^2 \ge \lambda_0 \int (\omega_\ell\phi)^2. \tag{3.14}$$

From (3.13) and (3.14), we get

$$\lambda_0 \int (\omega_\ell\phi)^2 \le \int \phi^2|d\omega_\ell|^2 - 2\int \rho\omega_\ell^2\phi^2 \le \lambda_0 \int \omega_\ell^2\phi^2 + \int \phi^2|d\omega_\ell|^2.$$

Now, if we let $\ell \to \infty$, then by the property $|d\omega_\ell| \le \frac{C}{\ell}$, we obtain

$$\lambda_0 \int \phi^2 \le -2\int \rho\phi^2 \le \lambda_0 \int \phi^2. \tag{3.15}$$

Under the assumption $\rho(x_0) > -\frac{1}{2}\lambda_0$ for some x_0, this implies that $\phi = 0$.

Now, we prove the second part. From the inequality $|2\langle a, b\rangle| \le t^2|a|^2 + \frac{1}{t^2}|b|^2$ for any $t \in \mathbb{R}$, we have

$$|2\int \omega\phi\langle d\phi, d\omega_\ell\rangle| \le t^2 \int \omega_\ell^2|d\phi|^2 + \frac{1}{t^2}\int \phi^2|d\omega_\ell|^2. \tag{3.16}$$

Comparing (3.12), (3.14) and (3.16), we obtain

$$\begin{aligned}(1-t^2)\int \omega_\ell^2|d\phi|^2 &\le -2\int \rho\omega_\ell^2\phi^2 + \frac{1}{t^2}\int \phi^2|d\omega_\ell|^2 \\ &\le \lambda_0 \int \omega_\ell^2\phi^2 + \frac{1}{t^2}\int \phi^2|d\omega_\ell|^2.\end{aligned}$$

Taking $t = \ell^{-\frac{1}{2}}$ and letting $\ell \to \infty$, the above inequality becomes

$$\int |d\phi|^2 \le -2\int \rho\phi^2 \le \lambda_0 \int \phi^2 \tag{3.17}$$

and hence $\phi \in S^1(M)$ (=the first sobolev space). Similarly from (3.16), we obtain the inequality

$$(1+t^2)\int \omega_\ell^2|d\phi|^2 \geq \int |d(\omega_\ell\phi)|^2 - (1+\frac{1}{t^2})\int \phi^2|d\omega_\ell|^2$$
$$\geq \lambda_0\int \omega_\ell^2\phi^2 - (1+\frac{1}{t^2})\int \phi^2|d\omega_\ell|^2.$$

Taking $t=\ell^{-\frac{1}{2}}$ and letting $\ell \to \infty$, we get

$$\int |d\phi|^2 \geq \lambda_0 \int \phi^2. \tag{3.18}$$

From (3.17) and (3.18), we have $\int |d\phi|^2 = \lambda_0 \int \phi^2$. Since λ_0 is the infimum of the spectrum of Δ^M, we have $\Delta^M\phi = \lambda_0\phi$ which implies that $\phi \in C^\infty(M)$. By maximum principle and $\phi \geq 0$, $\phi = 0$ or $\phi > 0$ everywhere. Assume $\phi \neq 0$. By Lemma 3.4 and our assumption, $\lambda_0\phi = \Delta^M\phi \leq -2\rho\phi \leq \lambda_0\phi$. That is, $\Delta^M\phi = -2\rho|s|$. This implies that $s=|s|s_1$ with $\nabla s_1 = 0$ everywhere and $\langle Rs_1, s_1\rangle = -\lambda_0$. Because $\Delta s_1 = \nabla s_1 = 0$, this implies that

$$-\lambda_0 = \langle Rs_1, s_1\rangle = \langle(\mathcal{R}+\bar{\mathcal{R}})s_1, s_1\rangle = 0$$

and hence ϕ is constant and $s_1 \in L^2(S)$. Hence we have that if $Vol(M) = +\infty$, then $\phi = 0$. □

Moreover, on $TM \subset \mathbb{C}l(M)$, we have ([8])

$$\mathcal{R}+\bar{\mathcal{R}} = \frac{1}{2}Ric.$$

Hence we have

COROLLARY 3.7. *On the complete Kähler manifold, if $Ric \geq -\lambda_0$ and $Ric > -\lambda_0$ at some point x_0, then every L^2−harmonic 1-form is necessarily zero.*

4. We shall consider some special cases of the results above. To begin, we suppose that M is a Kähler spin manifold, i.e., we assume that

there exists a principal Spin-bundle, $P_{Spin}(M) \to M$, with a $Spin_{2n}$-equivalent map $\tau : P_{Spin}(M) \to P_{SO}(M)$, to the bundle of real oriented orthonormal frame on M. The *bundle of spinors*, S, is then defined to be vector bundle associated to the unitary representation Δ of $Spin_{2n}$ given by the unique irreducible complex representation of Cl_{2n}, i.e., $S = P_{Spin} \times_\Delta \mathbb{C}^{2^n}$. This bundle is naturally a bundle of modules over $\mathbb{C}l(M)$ and carries a canonical connection induced from the lift of the riemannian connection on $P_{SO}(M)$. Since M is Kähler, this bundle S is naturally holomorphic and its connection is hermitian. On this bundle S, the curvature tensor R^S is given by

$$R^S_{V,W} = \frac{1}{4} \sum_{\alpha,\beta=1}^{2n} \langle R_{V,W} X_\alpha, X_\beta \rangle X_\alpha X_\beta, \tag{4.1}$$

where $X_1, \cdots, X_{2n}$ is any real orthonormal basis of the tangent space ([8]). Choosing a basis $e_1, \cdots, Je_n$, we can write R^S as

$$R^S_{V,W} = 2 \sum_{j,k=1}^{n} \langle R_{V,W} \xi_j, \bar{\xi}_k \rangle \bar{\xi}_j \xi_k + \sum_{j=1}^{n} \langle R_{V,W} \xi_j, \bar{\xi}_j \rangle.$$

Hence we have

$$\begin{aligned} \mathcal{R}^S &= \sum_{j,k=1}^{n} \xi_j \bar{\xi}_k R^S_{\bar{\xi}_j, \xi_k} \\ &= \sum_{i,j,k=1}^{n} \langle R_{\xi_i, \bar{\xi}_i} \bar{\xi}_j, \xi_k \rangle \xi_j \bar{\xi}_k \\ &= -\frac{1}{2} \sum_{j,k=1}^{n} Ric(\bar{\xi}_j, \xi_k) \xi_j \bar{\xi}_k, \end{aligned} \tag{4.2}$$

where Ric is Ricci tensor on M ([10]). Since Ric is hermitian symmetric, we may choose our basis so that $Ric(\bar{\xi}_j, \xi_k) = 1/2\lambda_j \delta_{jk}$, where $\lambda_j = Ric(e_j, e_j) = Ric(Je_j, Je_j)$, for $j = 1, \cdots, n$, are the eigenvalues. Then we have

$$\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D} = \nabla^* \nabla + \frac{1}{4} \sum_{j=1}^{n} \lambda_j \omega_j = \bar{\nabla}^* \bar{\nabla} + \frac{1}{4} \sum_{j=1}^{n} \lambda_j \bar{\omega}_j. \tag{4.3}$$

We note that $\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla} = \frac{1}{2}\tilde{\nabla}^*\tilde{\nabla}$ where

$$\tilde{\nabla}^*\tilde{\nabla} = -\sum_j (\nabla_{e_j,e_j} + \nabla_{Je_j,Je_j}) \tag{4.4}$$

is a self-adjoint, elliptic operator whose kernel is the space of parallel sections ([8]). We note that the scalar curvature κ of M is given by

$$\kappa = trace_R(Ric) = 2\sum_j \lambda_j. \tag{4.5}$$

Hence we get

THEOREM 4.1 ([10]). *On the spinor bundle S, we have*

$$4(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}) = \tilde{\nabla}^*\tilde{\nabla} + \frac{1}{4}\kappa,$$

where κ is the scalar curvature of M.

Summing up Theorem 3.3 and Theorem 4.1, we have

THEOREM 4.2. *Let M be a complete Kähler spin manifold. If $\kappa \geq -4\lambda_0$ for all $x \in M$ and either $\kappa > -4\lambda_0$ for some $x_0 \in M$ or (M, g) has infinite volume, then there are no non-trivial L^2-harmonic spinors.*

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