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A NOTE ON M-IDEALS OF COMPACT OPERATORS

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ABSTRACT. Suppose X is a closed subspace of $(\sum_{n=1}^{\infty} X_n)_{c_0}$, $\dim X_n < \infty$, which has the metric compact approximation property. It is proved that if Y is a Banach space of cotype q for some $2 \le q < \infty$ then $K(X,Y)$ is an M-ideal in $L(X,Y)$.

1. Introduction

A closed subspace J of a Banach space X is called an *M-ideal* in X if $J^{\perp} = \{x^* \in X^* : x^*(j) = 0 \text{ for all } j \in J\}$, the anihilator of J in X^*, is an L-summand in X^*. That is, there exists a closed subspace J' of X^* such that $X^* = J^{\perp} \oplus J'$, the algebraic direct sum of $J^{\perp}$ and J', and satisfies a norm condition $\|p+q\| = \|p\| + \|q\|$ whenever $p \in J^{\perp}$ and $q \in J'$.

Since the notion of an M-ideal in a Banach space was introduced by Alfsen and Effros [1], many authors have studied the problem determining those Banach spaces X and Y for which $K(X,Y)$, the space of compact linear operators from X to Y, is an M-ideal in $L(X,Y)$, the space of bounded linear operators from X to Y [2, 5, 6, 7, 9, 12, 13, 14]. It is well known that if X is a Hilbert space, ℓ_p $(1 < p < \infty)$ or c_0, then $K(X)(=K(X,X))$ is an M-ideal in $L(X)(=L(X,X))$ [5, 13] while $K(\ell_1)$ and $K(\ell_\infty)$ are not M-ideals in the corresponding space of operators [13]. Also several authors proved that $K(\ell_p, \ell_q)$ for $1 < p \le q < \infty$ is an M-ideal in $L(\ell_p, \ell_q)$ [4, 9, 12] and $K(X, c_0)$ is an M-ideal in $L(X, c_0)$ for every Banach space X [12, 13].

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In 1993, Kalton and Werner [7] established a necessary and sufficient condition for $K(X,Y)$ to be an M-ideal in $L(X,Y)$ for Banach spaces X and Y. More specifically, they proved the following theorem.

THEOREM 1 [7]. *Suppose that X is a Banach space such that there exists a sequence $\{K_n\}_{n=1}^{\infty}$ in $K(X)$ satisfying*

(i) $K_n \to I_X$ *strongly,*
(ii) $K_n^* \to I_{X^*}$ *strongly,*
(iii) $\|I_X - 2K_n\| \to 1$,

where I_X and I_{X^*} denote the identity maps on X and X^*, respectively. If Y is a Banach space, then $K(X,Y)$ is an M-ideal in $L(X,Y)$ if and only if every $T \in L(X,Y)$ with $\|T\| \leq 1$ has property (M).

According to Kalton and Werner [7] a continuous linear operator T with $\|T\| \leq 1$ from a Banach space X to a Banach space Y is said to have property (M) if

$$\limsup_{n\to\infty} \|y + Tx_n\| \leq \limsup_{n\to\infty} \|x + x_n\|$$

for all $x \in X$, $y \in Y$ with $\|y\| \leq \|x\|$ and all weakly null sequences $\{x_n\}_{n=1}^{\infty}$ in X.

In Theorem 2.2, using the Kalton-Werner criterion we will prove that if X is a closed subspace of $(\sum_{n=1}^{\infty} X_n)_{c_0}$, c_0-sum of finite dimensional Banach spaces X_n's, with the metric compact approximation property and Y is a Banach space of cotype $q \geq 2$ then $K(X,Y)$ is an M-ideal in $L(X,Y)$.

The c_0-sum $(\sum_{n=1}^{\infty} X_n)_{c_0}$ of Banach spaces X_n's is the Banach space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ ($x_n \in X_n$, $x_n \to 0$ as $n \to \infty$) with the norm $\|x\| = \sup_n \|x_n\|$. The ℓ_p-sum $(\sum_{n=1}^{\infty} X_n)_p$ is defined in an obvious fashion. We can easily see that $(\sum_{n=1}^{\infty} X_n)_{c_0}^* = (\sum_{n=1}^{\infty} X_n^*)_1$.

A Banach space X is said to have the metric compact approximation property if for every compact set K in X and every $\varepsilon > 0$, there is a compact linear operator $T : X \to X$ so that $\|T\| \leq 1$ and $\|Tx - x\| \leq \varepsilon$ for every $x \in K$. A Banach space X is said to be of cotype $q \geq 2$, if there is a constant $M < \infty$ so that for every finite set $\{x_j\}_{j=1}^{n}$ of vectors in X, we have

$$M \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| dt \geq \left(\sum_{j=1}^n \|x_j\|^q \right)^{1/q},$$

where r_j is the Rademarcher function on $[0,1]$.

2. M-ideals

LEMMA 2.1. *Let X be a closed subspace of $(\sum_{n=1}^\infty X_n)_{c_0}$ ($\dim X_n < \infty$). Then for all $x \in X$ and all weakly null sequence $\{x_k\}_{k=1}^\infty$ in X,*

$$\limsup_k \|x + x_k\| = \max\{\|x\|, \limsup_k \|x_k\|\}.$$

Proof. Let $x \in X$ and $\{x_k\}_{k=1}^\infty$ be a weakly null sequence in X. For each n, let J_n be the canonical injection of X_n into $(\sum_{n=1}^\infty X_n)_{c_0}$, P_n the canonical projection from $(\sum_{n=1}^\infty X_n)_{c_0}$ onto X_n and $Q_n = \sum_{j=1}^n P_j$. Then for each $x_n^* \in X_n^*$, $x_n^*(P_n x_k) = (J_n x_n^*)(x_k) \to 0$ as $k \to \infty$. Thus $P_n x_k \to 0$ as $k \to \infty$ weakly in X_n. Since $\dim X_n < \infty$, $\|P_n x_k\| \to 0$ as $k \to \infty$ and hence $\|Q_n x_k\| \to 0$ as $k \to \infty$ for every n. Since

$$\begin{aligned} \|x + x_k\| &\leq \|Q_n x + (I - Q_n) x_k\| + \|(I - Q_n)x\| + \|Q_n x_k\| \\ &\leq \max\{\|x\|, \|x_k\|\} + \|(I - Q_n)x\| + \|Q_n x_k\|, \end{aligned}$$

taking $\limsup_k$ and then letting $n \to \infty$, we get that

$$\limsup_k \|x + x_k\| \leq \max\{\|x\|, \limsup_k \|x_k\|\}.$$

Similarly, we can get the reversed inequality from inequality

$$\|x + x_k\| \geq \|Q_n x + (I - Q_n) x_k\| - \|(I - Q_n)x\| - \|Q_n x_k\|. \qquad \square$$

THEOREM 2.2. *If X is a closed subspace of $(\sum_{n=1}^\infty X_n)_{c_0}$ ($\dim X_n < \infty$) with the metric compact approximation property and Y is a Banach space of cotype q ($2 \leq q < \infty$), then $K(X, Y)$ is an M-ideal in $L(X, Y)$.*

Proof. Since X is a closed subspace of $(\sum_{n=1}^{\infty} X_n)_{c_0}$ with the metric compact approximation property, $K(X)$ is an M-ideal in $L(X)$ [14], and hence [4, p. 299] there is a sequence $\{K_n\}_{n=1}^{\infty}$ in $K(X)$ such that

(i) $K_n \to I_X$ strongly,
(ii) $K_n^* \to I_{X^*}$ strongly,
(iii) $\|I_X - 2K_n\| \to 1$ as $n \to \infty$.

Therefore, by Theorem 1 it suffices to show that every $T \in L(X, Y)$ with $\|T\| \leq 1$ has the property (M).

Let $x \in X$, $y \in Y$ with $\|y\| \leq \|x\|$, and let $\{x_n\}_{n=1}^{\infty}$ be a weakly null sequence in X. In view of Lemma 2.1 we only need to prove that $\|Tx_n\| \to 0$. Suppose $\|Tx_n\| \not\to 0$. By passing to a subsequence we may assume that $\|Tx_n\| \geq \delta > 0$ for all n. By the gliding hump argument used in [10, Proposition 1.a.12], we can get a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of c_0. We also may assume that $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ for c_0. Hence, there exists $\alpha > 0$ such that $\left\|\sum_{n=1}^{N} \pm e_n\right\| \geq \alpha \left\|\sum_{n=1}^{N} \pm x_n\right\|$ for all N and all choices of signs.

Since $\|T\| = \|T\| \left\|\sum_{n=1}^{N} \pm e_n\right\| \geq \alpha \left\|\sum_{n=1}^{N} \pm Tx_n\right\|$ for all N and all choices of signs, we have

$$\begin{aligned} \|T\| &\geq \alpha \ \operatorname*{Ave}_{\pm} \left\| \sum_{n=1}^{N} \pm Tx_n \right\| \\ &= \alpha \int_0^1 \left\| \sum_{n=1}^{N} r_n(t) Tx_n \right\| dt \\ &\geq \alpha\, M^{-1} \left(\sum_{n=1}^{N} \|Tx_n\|^q \right)^{1/q} \\ &\geq \alpha\, M^{-1}\, N^{1/q}\, \delta \qquad \text{for all } N, \end{aligned}$$

which is a contradiction. Hence $\|Tx_n\| \to 0$ and the proof is complete.□

Since $L_p(0,1)$ $(1 \leq p < \infty)$ is of cotype $\max\{2, p\}$ [11, p. 73], we have;

COROLLARY 2.3. *If X is as in* Theorem 2.2 *and* $1 \leq p < \infty$, *then* $K(X, L_p(0,1))$ *is an M-ideal in* $L(X, L_p(0,1))$.

REMARK. We can not allow $q = \infty$ in Theorem 2.2. c_0 and ℓ_∞ are of cotype ∞. $K(c_0)$ is an M-ideal in $L(c_0)$, while $K(c_0, \ell_\infty)$ is not an M-ideal in $L(c_0, \ell_\infty)$ [12].

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