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REMARK ON GENERALIZED k-QUASIHYPONORMAL OPERATORS

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ABSTRACT. An operator $T \in \mathcal{L}(H)$ is generalized k-quasihyponormal if there exists a constant $M > 0$ such that

$$T^{*k}[M^2(T-z)^*(T-z)-(T-z)(T-z)^*]T^k \geq 0$$

for some integer $k \geq 0$ and all $z \in \mathbf{C}$. In this paper, we show that if T is a generalized k-quasihyponormal operator with the property $0 \notin \sigma(T)$, then T is subscalar of order 2. As a corollary, we get that such a T has a nontrivial invariant subspace if its spectrum has interior in $\mathbf{C}$.

1. Introduction

Let H and K be separable complex Hilbert spaces and $\mathcal{L}(H,K)$ denote the space of all bounded linear operators from H to K. If $H = K$, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H,K)$.

An operator T is called *hyponormal* if $T^*T \geq TT^*$, or equivalently, if $\|Th\| \geq \|T^*h\|$ for all $h \in H$. A larger class of operators related to hyponormal operators is the following: $T \in \mathcal{L}(H)$ is called *M-hyponormal* if there exists a constant M such that $\|(T-z)^*h\| \leq M\|(T-z)h\|$ for all $h \in H$ and all $z \in \mathbf{C}$. There are classical examples of an M-hyponormal, non-hyponormal operator (see [6]).

We now define a *generalized k-quasihyponormal* operator introduced and studied by Huang (see [3]): $T \in \mathcal{L}(H)$ is a generalized k-quasihyponormal operator (i.e., $T \in GQ(k)$) if there exists a constant M such

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that

$$T^{*k}[M^2(T-z)^*(T-z)-(T-z)(T-z)^*]T^k \geq 0,$$

or equivalently,

$$M\|(T-z)T^k h\| \geq \|(T-z)^*T^k h\|,$$

for some integer $k \geq 0$ and all $z \in \mathbf{C}$. These operators generalize M-hyponormal ones and k-quasihyponormal ones. (Recall that M-hyponormal operators are the ones with $k=0$ and k-quasihyponormal operators are the ones with $M=1$ in this paper.)

A bounded linear operator S on H is called *scalar* of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism,

$$\Phi : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(H)$$

such that $\Phi(z) = S$, where z stands for the identity function on $\mathbf{C}$ and $C_0^m(\mathbf{C})$ for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m, $0 \leq m \leq \infty$. An operator is *subscalar* if it is similar to the restriction of a scalar operator to an invariant subspace.

This paper has been divided into three sections. Section two deals with some preliminary facts. In section three, we shall prove our main results.

2. Preliminaries

An operator $T \in \mathcal{L}(H)$ is said to satisfy the *single valued extension property* if for any open subset U in $\mathbf{C}$, the function

$$z - T : \mathcal{O}(U,H) \longrightarrow \mathcal{O}(U,H)$$

defined by the usual pointwise multiplication is one-to-one where $\mathcal{O}(U,H)$ denotes the Frechet space of H-valued analytic functions in U with respect to uniform topology. If, in addition, the above function $z-T$ has closed range on $\mathcal{O}(U,H)$, then T satisfies the *Bishop's condition* (β).

Let z be the coordinate in $\mathbf{C}$ and let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Fix a separable complex Hilbert space H and a bounded (connected) open subset U of $\mathbf{C}$. We shall denote by $L^2(U,H)$ the Hilbert space of measurable functions $f : U \to H$, such that

$$\|f\|_{2,U} = \left\{ \int_U \|f(z)\|^2 d\mu(z) \right\}^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(U,H)$ which are analytic functions in U (i.e., $\bar{\partial} f = 0$) is denoted by

$$A^2(U,H) = L^2(U,H) \cap \mathcal{O}(U,H).$$

The set $A^2(U,H)$ is called the *Bergman space* for U. Note that $A^2(U,H)$ is a Hilbert space. We denote by P the orthogonal projection of $L^2(U,H)$ onto $A^2(U,H)$.

Let us define now a special Sobolev type space. Let U be again a bounded open subset of $\mathbf{C}$ and m be a fixed non-negative integer. The *vector valued Sobolev space* $W^m(U,H)$ with respect to $\bar{\partial}$ and of order m will be the space of those functions $f \in L^2(U,H)$ whose derivatives $\bar{\partial} f, \cdots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U,H)$. Endowed with the norm

$$\|f\|_{W^m}^2 = \sum_{i=0}^{m} \|\bar{\partial}^i f\|_{2,U}^2$$

$W^m(U,H)$ becomes a Hilbert space contained continuously in $L^2(U,H)$.

We next discuss the fact concerning the multiplication operator by z on $W^m(U,H)$. The linear operator M of multiplication by z on $W^m(U,H)$ is continuous and it has a spectral distribution of order m, defined by the relation

$$\Phi_M : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(W^m(U,H)), \quad \Phi_M(f) = M_f.$$

Therefore, M is a scalar operator of order m by the definition given in Section one.

3. Main results

This section deals with the characterization for some generalized k-quasihyponormal operators. We need the following lemmas to give a proof of the main theorem.

LEMMA 1 ([5], Proposition 2.1). *For every bounded disk D in $\mathbf{C}$, there is a constant C_D such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D,H)$ we have*

$$\|(I-P)f\|_{2,D} \leq C_D(\|(T-z)^*\bar{\partial}f\|_{2,D} + \|(T-z)^*\bar{\partial}^2 f\|_{2,D}),$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space $A^2(D,H)$.

COROLLARY 2. *Let D and C_D be as in* Lemma 1. *If $T \in \mathcal{L}(H)$ is a generalized k-quasihyponormal operator and $f \in W^2(D,H)$, then for some integer $k \geq 0$ and all $z \in \mathbf{C}$, there exists a constant M such that*

$$\|(I-P)T^k f\|_{2,D} \leq MC_D(\|(z-T)\bar{\partial}(T^k f)\|_{2,D} + \|(z-T)\bar{\partial}^2(T^k f)\|_{2,D})$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto $A^2(D,H)$.

Proof. This follows from Lemma 1 and the definition of a generalized k-quasihyponormal operator. □

LEMMA 3. *Let T be a generalized k-quasihyponormal operator with the property $0 \notin \sigma(T)$ and let D be a bounded disk which contains $\sigma(T)$. Then the operator $V : H \to H(D)$ defined by*

$$Vh = 1 \otimes h + \overline{(z-T)W^2(D,H)} \ (= \widetilde{1 \otimes h})$$

is one-to-one and has a closed range, where $H(D) = W^2(D,H)/\overline{(z-T)W^2(D,H)}$ and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h.

Proof. Let $h_i \in H$ and $f_i \in W^2(D,H)$ be sequences such that

$$\lim_{i\to\infty} \|(z-T)f_i + 1 \otimes h_i\|_{W^2} = 0. \tag{1}$$

Then by the definition of the norm of Sobolev space, (1) implies

$$\lim_{i\to\infty}(\|(z-T)\bar{\partial}f_i\|_{2,D} + \|(z-T)\bar{\partial}^2 f_i\|_{2,D}) = 0. \tag{2}$$

Since T is a generalized k-quasihyponormal operator, Corollary 2 and the equation (2) imply

$$\lim_{i\to\infty} \|(I-P)T^k f_i\|_{2,D} = 0, \tag{3}$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto $A^2(D, H)$. Now it follows from (1) that

$$\lim_{i\to\infty} \|(z-T)T^k f_i + 1 \otimes T^k h_i\|_{2,D} = 0. \tag{4}$$

By (3) and (4), we have

$$\lim_{i\to\infty} \|(z-T)PT^k f_i + 1 \otimes T^k h_i\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{i\to\infty} \|PT^k f_i(z) + (z-T)^{-1}(1 \otimes T^k h_i)\| = 0$$

uniformly. Hence, by Riesz functional calculus,

$$\lim_{i\to\infty} \|\frac{1}{2\pi i}\int_\Gamma PT^k f_i(z)dz + T^k h_i\| = 0.$$

But $\int_\Gamma PT^k f_i(z)dz = 0$ by Cauchy's theorem. Hence,

$$\lim_{i\to\infty} T^k h_i = 0. \tag{5}$$

Since $0 \notin \sigma(T)$, $\lim_{i\to\infty} h_i = 0$. □

REMARK 4. If $k = 0$, it is easily observed that the condition $0 \notin \sigma(T)$ in Lemma 3 is not necessary from (5).

THEOREM 5. *Let $T \in \mathcal{L}(H)$ be a generalized k-quasihyponormal operator. If $0 \notin \sigma(T)$, then T is subscalar of order 2.*

Proof. Consider an arbitrary bounded open disk D in $\mathbf{C}$ which contains $\sigma(T)$ and the quotient space

$$H(D) = W^2(D, H)/\overline{(z-T)W^2(D, H)}$$

endowed with the Hilbert space norm. The class of a vector f or an operator on $H(D)$ will be denoted by $\tilde{f}$, respectively $\tilde{A}$. Let $M(= M_z)$ be the multiplication operator by z on $W^2(D, H)$. Then M is a scalar operator of order 2 and its spectral distribution is

$$\Phi : C_0^2(\mathbf{C}) \longrightarrow \mathcal{L}(W^2(D, H)), \quad \Phi(f) = M_f,$$

where M_f is the multiplication operator with f. Since M commutes with $z - T$, $\tilde{M}$ on $H(D)$ is still a scalar operator of order 2, with $\tilde{\Phi}$ as a spectral distribution.

Let V be the operator

$$Vh = \widetilde{1 \otimes h}(= 1 \otimes h + \overline{(z-T)W^2(D,H)}),$$

from H into $H(D)$, denoting by $1 \otimes h$ the constant function h. Then $VT = \tilde{M}V$. Since V is one-to-one and has a closed range by Lemma 3, ran V is a closed invariant subspace for the scalar operator $\tilde{M}$. Therefore, T is subscalar of order 2. □

REMARK 6. If $k = 0$ in Theorem 5, Remark 4 implies that the condition $0 \notin \sigma(T)$ in Theorem 5 is not necessary.

COROLLARY 7. *Let $T \in \mathcal{L}(H)$ be a generalized k-quasihyponormal operator. If $0 \notin \sigma(T)$ and $\sigma(T)$ has interior in the complex plane $\mathbf{C}$, then T has a nontrivial invariant subspace.*

Proof. This follows from Theorem 5 and [2]. □

COROLLARY 8. *If $T \in \mathcal{L}(H)$ is an invertible generalized k-quasihyponormal operator, then T has the property (β).*

Proof. Since the property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 5 to the case of a scalar operator of order 2. Since every scalar operator has the property (β) (see [5]), T has the property (β). □

COROLLARY 9. Any M-hyponormal operator is subscalar of order 2.

Proof. It is clear from Remark 6. □

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