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ASYMPTOTIC LENS EQUIVALENCE IN MANIFOLDS WITHOUT CONJUGATE POINTS

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ABSTRACT. We prove the asymptotic lens equivalence in manifolds without conjugate points. By using this property we show that under a metric condition of asymptotically Euclidean and the curvature condition decaying faster than quadratic, any surface (R^2, g) without conjugate points is Euclidean.

1. Introduction

A complete Riemannian manifold has no conjugate points if any two points in its universal cover are joined by a unique geodesic. This condition is a natural generalization of nonpositive curvature. Hence many results for manifolds of nonpositive curvature can be generalized to manifolds without conjugate points.

In [8] they showed that the Euclidean spaces R^n $(n \geq 3)$ are rigid under compactly supported perturbations with nonpositive (or nonnegative) sectional curvature by using the generalized Gauss-Bonnet theorem. Also these results were generalized to an asymptotic condition in [9].

In the case of manifolds without conjugate points we can not use the curvature condition. However [7, 2, 3] established that the Euclidean space is rigid under compactly supported perturbations having no conjugate points by using the Hopf theorem and the lens equivalence. The lens equivalence of the two metrics g_0, g_1 means that if γ_0 is a g_0-geodesic such that $\gamma_0(t)$ does not lie in some compact set K for

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$|t| > T$ then there is a g_1-geodesic γ_1 such that $\gamma_0(t) = \gamma_1(t)$ for all $|t| > T$. This implies that for any $x, y \in \partial K$ $d_0(x,y) = d_1(x,y)$.

The purpose of this paper is to develop the basic tool of the proof of a rigidity theorem in the manifold without conjugate points, i.e., the asymptotic lens equivalence which extends the condition of compactly supported perturbations to an asymptotically decaying metric condition.

Let (R^n, g_1) be a manifold without conjugate points, and g_0 be the Euclidean metric in R^n. The asymptotic lens equivalence of two metrics g_0, g_1 means that for any geodesic $\gamma(t)$ in g_1 metric there exists a unique line $l(t)$, i.e., a geodesic with respect to (R^n, g_0) such that the distance of $\gamma(t)$ and $l(t)$ decrease as t goes to infinity.

In this paper (R^n, g_1) is called "asymptotically Euclidean" if for some fixed point $p \in R^n$ and a number $\alpha(0 < \alpha \leq 1)$, we have

$$|g_0(x) - g_1(x)| < Cr(x)^{-2-\alpha}, \quad |\partial_i g_1(x)| < Cr(x)^{-3-\alpha},$$

where C is a constant and $r(x)$ is the distance from p to x with respect to g_0.

In this paper we will prove the following asymptotic lens equivalence.

ASYMPTOTIC LENS EQUIVALENCE. *If (R^n, g_1) is a manifold without conjugate points with an asymptotically Euclidean metric, then (R^n, g_1) and (R^n, g_0) satisfy the asymptotic lens equivalence.*

As an application we show that any surface without conjugate points which has an asymptotically Euclidean metric and a curvature decaying faster than quadratic is Euclidean. In order to prove this rigidity theorem, we first show that there is an 1-1 correspondence of geodesics between (M^2, g_1) and the Euclidean plane by using the asymptotic lens equivalence. Then we prove the weak parallelism of (R^2, g_1) in the sense of [1]. This implies the rigidity theorem by [1]. In [11] they showed this rigidity theorem by another method. Unfortunately this method can not be extended to higher dimensional cases. Since we already obtained the asymptotic lens equivalence in any dimension, if another ingredients such as [3, 4, 5] is established we can prove the following rigidity conjecture.

CONJECTURE. *If (R^n, g_1) is a manifold without conjugate points with an asymptotically Euclidean metric and a curvature decaying faster than quadratic, then (R^n, g_1) is Euclidean.*

This work began while the author was visiting the University of Pennsylvania, and he thanks all the faculty members in the Mathematics Department, especially C. Croke for introducing this problem and discussing the result.

2. Basic properties

In this section we will first consider the relation between the distance functions of (R^n, g_1) and (R^n, g_0). Throughout this paper, the lower index 1 means the given asymptotically Euclidean space and the lower index 0 means an Euclidean space.

PROPOSITION 1. *For any points $x, y \in R^n$, let $\gamma_i(x, y)$ be a geodesic segment between x and y in (R^n, g_i). If $\gamma_i(x, y) \cap B_0(p, R_1) = \phi$ for some number $R_1 > 1$ and any i, then*

$$|d_0(x, y) - d_1(x, y)| < C_1 {R_1}^{-1-\alpha},$$

where C_1 is a constant which is independent of the choice of x, y.

Proof. Put $l_i = d_i(x, y)$. Assume that γ_i has the arclength parameter with respect to (R^n, g_i) such that $\gamma_i(0) = x \in \partial B_0(p, R_1)$ and $\gamma_i(l_i) = y$. Using the Cauchy-Schwartz inequality, we have

$$(d_j(x, y))^2 \leq \left(\int_0^{l_i} |\gamma_i'|_j dt\right)^2 \leq l_i \int_0^{l_i} |\gamma_i'(t)|_j^2 dt,$$

where $|\cdot|_i$ denotes the norm defined by g_i. Since $|\gamma_i'|_i = 1$,

$$(d_i(x, y))^2 = l_i^2 = l_i \int_0^{l_i} |\gamma_i'(t)|_i^2 dt.$$

Hence by the asymptotically Euclidean property

$$(d_j(x, y))^2 - (d_i(x, y))^2 \leq l_i C_2 \int_0^{l_i} r(\gamma_i(t))^{-2-\alpha} dt.$$

When $i = 0$, $j = 1$ and $t < 8R_1$, since $r(\gamma_0(t)) > R_1$

$$\int_0^{l_0} r(\gamma_0(t))^{-2-\alpha} dt \leq 8R_1(R_1)^{-2-\alpha} = 8{R_1}^{-1-\alpha}.$$

If $t \geq 8R_1$, the triangle inequality for triangle $(px\gamma_0(t))$ yields

$$\begin{aligned} r(\gamma_0(t)) = d_0(p, \gamma_0(t)) &\geq (t - R_1) \\ &\geq (\frac{t}{2} + 4R_1 - R_1) = \frac{t}{2} + 3R_1. \end{aligned}$$

Hence

$$\begin{aligned} l_1^2 - l_0^2 \leq & l_0 \left(\int_0^{8R_1} r(\gamma_0(t))^{-2-\alpha} dt + 3C_2 \int_{8R_1}^{l_0} (\frac{t}{6} + R_1)^{-2-\alpha} dt \right) \\ \leq & l_0 C_3 {R_1}^{-1-\alpha}. \end{aligned}$$

Therefore

$$(l_1 - l_0) \leq C_3 {R_1}^{-1-\alpha}.$$

We conclude that

$$d_1(x, y) \leq d_0(x, y) + C_3 {R_1}^{-1-\alpha}.$$

When $i = 1$, $j = 0$, by the previous result

$$\begin{aligned} t = d_1(x, \gamma_1(t)) &\leq d_0(x, \gamma_1(t)) + C_3 {R_1}^{-1-\alpha} \\ &\leq d_0(p, \gamma_1(t)) + d_0(p, x) + C_3 {R_1}^{-1-\alpha}. \end{aligned}$$

If $t > 8R_1$, then we have

$$\begin{aligned} r(\gamma_1(t)) = d_0(p, \gamma_1(t)) &\geq t - R_1 - C_3 {R_1}^{-1-\alpha} \\ &\geq t - 2R_1 \\ &\geq \frac{t}{2} + 4R_1 - 2R_1 \\ &\geq \frac{t}{2} + 2R_1. \end{aligned}$$

Hence by a similar argument, we obtain

$$d_0(x, y) \leq d_1(x, y) + C_3 {R_1}^{-1-\alpha}.$$

□

Next we will investigate the asymptotic behavior of geodesics by using the first derivative condition of the metric.

LEMMA 1. *Let $\gamma_1(t)$ be a geodesic ray of (R^n, g_1) which lies outside $B_0(p, R_1)$, κ_0 be the geodesic curvature with respect to (R^n, g_0). Then for some constant C_4*

$$|\kappa_0(\gamma_1(t))| \leq C_4 r(\gamma_1(t))^{-3-\alpha}.$$

Proof. Let $\gamma_1'(t) = \sum v^i X_i$, where $\{X_i\}$ is an orthonormal basis with respect to g_0. Since γ_1 is a geodesic in (R^n, g_1), it satisfies

$$\frac{D\gamma_1'}{dt} = \sum_k \left\{ \frac{dv^k}{dt} + \sum_{ij} v^i v^j \Gamma_1{}_{ij}^k \right\} X_k = 0$$

where $\Gamma_1{}_{ij}^k$ is the Christoffel symbol of (R^n, g_1). Since the Christoffel symbol is expressed by the sum of 1-st derivatives of metric, i.e.,

$$\Gamma_{ij}^k = \frac{1}{2} \sum_m \left\{ \frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{mi}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m} \right\} g^{km},$$

it satisfies that

$$|\Gamma_1{}_{ij}^k(x)| \leq C_4 r(x)^{-3-\alpha}.$$

Assume that $\gamma_1(t)$ has the arclength parametrization with respect to (R^n, g_0). Then

$$\begin{aligned} |\kappa_0(\gamma_1(t))| &= \left| \frac{D_0 \gamma_1'(t)}{dt} \right| \\ &= \left| \sum_k \frac{dv^k}{dt} X_k \right| \\ &= \sum_k \left| \sum_{i,j} v^j v^i \Gamma_1{}_{ij}^k \right| \\ &\leq C_4 r(\gamma_1(t))^{-3-\alpha}. \end{aligned}$$

□

PROPOSITION 2. *Let $\gamma_1(t)$ be a geodesic ray in (R^n, g_1) such that $\gamma_1(0) = q$ lies in $\partial B_0(p, R_1)$ and $\gamma_1'(0) = v$ is a radial direction of $B_0(p, R_1)$. Then $\gamma_1(t)$ $(t > 0)$ lies in the cone $Cone(q, v)$ with the vertex q and angle $\theta = C_5 {R_1}^{-2-\alpha}$ from v where C_5 is some constant independent q.*

Proof. In any point of $\gamma_1(t)$, $\theta(t)$ has the maximum value when $\gamma_1'(t)$ lies in the plane P which passes through $\gamma_1(t)$ and v. Therefore we will consider $\gamma_1(t)$ which lies in P with the polar coordinate (r, θ).

When $\theta = f(r)$, the geodesic curvature $\kappa_0(\gamma_1(t))$ is

$$\frac{|2f'(r) + rf''(r) + r^2(f'(r))^3|}{|1 + r^2(f'(r))^2|^{3/2}}.$$

Using Lemma 1, $f(r)$ must decrease with the order of $r^{-2-\alpha}$, i.e.,

$$\theta < C_5 {R_1}^{-2-\alpha}.$$

By this proposition, we know that no geodesic turns around p for a sufficiently large R_1. Furthermore when σ is the line from $\gamma_1(0)$ with the direction $\gamma_1'(0)$, the height function $h(t)$ from σ is the integral of $\sin\theta(t) \sim \theta(t)$,

$$h(t) \le C_6 {R_1}^{-1-\alpha}.$$

Therefore any geodesic γ_1 will become straight. □

PROPOSITION 3. *For any sufficiently large R_2 let $\gamma_1(t)$ be a ray such that it lies outside $B_0(p, R_2)$ with $\gamma_1(0) \in \partial B_0(p, R_2)$ and $\gamma_1'(0)$ is not tangent to $\partial B_0(p, R_2)$. Then there exists a unique Euclidean line l such that*

$$d_0(\gamma_1(t), l) < C_7 {R_2}^{-1-\alpha} (t > 0).$$

Furthermore we can take the parameter such that

$$d_0(\gamma_1(t), l(t)) < C_7 {R_2}^{-1-\alpha} (t > 0).$$

Proof. We can assume that $Cone(\gamma_1(0), \gamma_1'(0)) \cap B_0(p, R_2) = \phi$ since $\gamma_1'(0)$ is not tangent to $\partial B_0(p, R_2)$.

Let P_1 be a tangent plane to the $Cone(\gamma_1(0), \gamma_1'(0))$ and $\tilde{\gamma}_1$ be the projection of γ_1 to the plane P_1. Consider the line l_t from $\tilde{\gamma}_1(0)$ to $\tilde{\gamma}_1(t)$, let

$$\limsup_{t\to\infty} l_t'(0) = v_s, \quad \liminf_{t\to\infty} l_t'(0) = v_i.$$

Since $l_t'(0)$ is bounded by proposition 2, we can find v_s, v_i.

If $v_s \neq v_i$, we can take the line $\sigma(t)$ such that $\sigma(0) = \tilde{\gamma}_1(0)$ and $\sigma'(0) = v_m (v_i < v_m < v_s)$. By Lemma 2 the height of $\gamma_1(t)$ from σ decreases and v_s and v_i must be same. Call this direction v_1.

We can find the tangent planes P_k $(k = 1, \cdots, n)$ which are not parallel to each other, and the direction v_k with respect to plane P_k. Therefore we have a unique asymptotic direction v. Also we can take the circular cylinder with the axis v such that $\gamma_1(t)$ $(t \geq 0)$ is contained in this cylinder and its diameter is the smallest. Since the height $h(t)$ is the radius of the cylinder and it decreases as r increases, we can find a unique asymptotic line l such that for any $t > 0$, R_2

$$l'(0) = v, \quad d_0(\gamma_1(t), l) < C_7 {R_2}^{-1-\alpha}.$$

□

3. Proof of asymptotic lens equivalence

We will prove the asymptotic lens equivalence, and this property implies that there is an one to one correspondence between g_1 geodesic and g_0 geodesic.

PROPOSITION 4 (Asymptotic lens equivalence). *For any geodesic γ in (R^n, g_1) there exists exactly one line l such that $\gamma(t)$ is asymptotic to $l(t)$ as $|t|$ goes to infinity. And for any $\tilde{x}, \tilde{y} \in \partial B_0(p, R_3)$ and sufficiently large R_3,*

$$|d_0(\tilde{x}, \tilde{y}) - d_1(x, y)| < C_8 {R_3}^{-1-\alpha}$$

Proof. Let γ be a geodesic from $\tilde{x}$ to $\tilde{y}$ in (R^n, g_1) such that $\gamma(0) = \tilde{x}, \gamma(a) = \tilde{y}$ for any $\tilde{x}, \tilde{y} \in \partial B_0(p, R_3)$. By Proposition 3, there exist two asymptotic lines $l_1(t), l_2(t)$ such that

$$d_0(\gamma(t), l_1(t)) < C_7 {R_3}^{-1-\alpha} \quad \text{for} \quad t < 0,$$
$$d_0(\gamma(t), l_2(t)) < C_7 {R_3}^{-1-\alpha} \quad \text{for} \quad t > a.$$

Then we can easily show that l_1 and l_2 have the same direction. If not, for a large t there exists a minimal geodesic in (R^n, g_1) through $l_1(-t), l_2(t)$ which does not pass $B_0(p, R_3)$ by proposition 1. This is a contradiction to the assumption of no conjugate points.

Let $x = \partial B_0(p, R_3) \cup l_1(t)$ for $t < 0$ and $y = \partial B_0(p, R_3) \cup l_2(t)$ for $t > a$. If these x, y are not found, then we should take a little bigger ball. Also we assume that $\angle(u\tilde{x}x) \leq \pi/3$ and $\angle(w\tilde{y}y) \leq \pi/3$, where u is the foot of the perpendicular line from $\tilde{x}$ to l_1 and w is the foot of the perpendicular line from $\tilde{y}$ to l_2.

We will first show that $d_1(\tilde{x}, \tilde{y}) \leq d_0(x, y) + \epsilon$. Choose any point z in $R^n - B_0(p, R_3)$. We thus have

$$\begin{aligned} 2t + d_1(\tilde{x}, \tilde{y}) &= d_1(\gamma(-t), \gamma(t+a)) \\ &\leq d_1(\gamma(-t), z) + d_1(\gamma(t+a), z)). \end{aligned}$$

For large values of t and appropriate choice of z, g_0 geodesic segments from z to $l_i(t)$ do not intersect $B_0(p, R_3)$. By Proposition 1,

$$d_1(\tilde{x}, \tilde{y}) \leq d_0(l_1(-t), z) + d_0(l_2(t+a), z) - 2t + 2C_7 {R_3}^{-1-\alpha}.$$

As t goes to infinity, we get

$$\begin{aligned} d_1(\tilde{x}, \tilde{y}) &\leq d_0^{\pi}(x, y) + 2C_7 {R_3}^{-1-\alpha} \\ &\leq d_0(x, y) + 2C_7 {R_3}^{-1-\alpha}, \end{aligned}$$

where d_0^{π} is the length of the projection of line segment to l_i.

Now let $l(t)$ be a line in R^n parameterized by arclength which dose not pass through $B_0(p, R_3)$ but is parallel to and has the same orientation as the line from x to y. Then

$$\begin{aligned}
2t =& d_0(l(-t), l(t)) \\
\leq& d_1(l(-t), l(t)) + C_1 {R_3}^{-2-\alpha} \\
\leq& d_1(l(-t), \tilde{x}) + d_1(\tilde{x}, \tilde{y}) + d_1(\tilde{y}, l(t)) + C_1 {R_3}^{-2-\alpha} \\
\leq& d_0(l(-t), \tilde{x}) + d_1(\tilde{x}, \tilde{y}) + d_0(\tilde{y}, l(t)) + 3C_1 {R_3}^{-2-\alpha} \\
\leq& d_0(l(-t), x) + d_0(x, \tilde{x}) + d_1(\tilde{x}, \tilde{y}) + d_0(y, l(t)) + d_0(y, \tilde{y}) + 3C_1 {R_3}^{-1-\alpha} \\
\leq& d_0(l(-t), x) + d_1(\tilde{x}, \tilde{y}) + d_0(y, l(t)) + 7C_7 {R_3}^{-1-\alpha}.
\end{aligned}$$

Thus we see that

$$2t - d_0(l(-t), x) - d_0(l(t), y) \leq d_1(\tilde{x}, \tilde{y}) + 7C_7 {R_3}^{-1-\alpha}.$$

Taking the limit as t goes to infinity, we get

$$d_0(x, y) \leq d_1(\tilde{x}, \tilde{y}) + 7C_7 {R_3}^{-1-\alpha}.$$

Finally we get

$$d_0(x, y) \leq d_1(\tilde{x}, \tilde{y}) + 7C_7 {R_3}^{-1-\alpha} \leq d_0^{\pi}(x, y) + 9C_7 {R_3}^{-1-\alpha}.$$

Therefore

$$|d_0(x, y) - d_1(x, y)| < C_8 {R_3}^{-1-\alpha}.$$

If two asymptotic lines are not same, $\Delta = d_0(x, y) - d_0^{\pi}(x, y)$ is a strictly positive number. Therefore

$$d_0(x, y) \leq d_0(x, y) - \Delta + 7C_7 {R_3}^{-1-\alpha}.$$

However we can choose two points $x \in l_1$, $y \in l_2$ such that $-\Delta + 7C{R_1}^{-1-\alpha} < 0$ since Δ decreases as the polynomial degree of order 1 by the direct calculation in the Euclidean space. Hence for any geodesic in (R^n, g_1), two asymptotic lines are same. □

4. Application of the asymptotic lens equivalence

Let M be a 2-dimensional asymptotically Euclidean plane without conjugate points which has the quadratic decaying condition of curvature, i.e.,

$$|K(x)| < Cr(x)^{-2-\alpha}(0 < \alpha \leq 1).$$

In this section we will show that M is Euclidean.

First we show that the set of g_1 geodesics has a one to one correspondence to the set of g_2 geodesics using the asymptotic lens equivalence. Consider the scalar Jacobi equation,

$$y'' + K(x)y = 0,$$

where $K(x)$ is the curvature function of (M, g_1). In [10], it is show that if $\int_1^\infty xK(x)$ is finite, then there is a solution $y_1(x)$ such that

$$\lim_{x\to\infty} y_1(x) = 1,$$

and its general solution is of the form

$$y(x) = A(1 + f(x)) + Bx(1 + g(x)),$$

where A and B are arbitrary constants and $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to\infty} g(x) = 0$. Since in our case the curvature function decays faster than quadratic order, every solution of the Jacobi equation is of the form

$$y(x) = A(1 + f(x)) + Bx(1 + g(x)).$$

Hence every stable or unstable Jacobi field $y_\pm$ has a form

$$y_\pm = A(1 + f(x)),$$

where $1 + f(x) > 0$.

Suppose two g_1 geodesics have the same asymptotic line. Since (M, g_1) has no conjugate points, two geodesics have at most one intersection point.

i) If two geodesic γ_1, γ_2 have one intersection point q, by [6] we can show that this is impossible; If a curvature is bounded below, then geodesic rays from any point are uniformly divergent. Hence any two geodesics intersect at q are divergent.

ii) If γ_1, γ_2 have no intersection points, there is a Jacobi field $J(x)$ such that

$$\lim_{x \to \pm\infty} J(x) = 0.$$

However this is impossible by above remark.

The remaining thing is the surjectivity. That is, for any line l_1 we must find a g_1 geodesic which is asymptotic to l_1. For some fixed point p_1, we can find minimal g_1-geodesic $\gamma_x(t)$ from p_1 to $l_1(x)$. As t goes to infinity, there is a limit g_1-geodesic $\gamma_\infty(t)$ which has an asymptotic line l_2 parallel to l_1. If l_2 lies in the half plane lying p_1 with respect to l_1, then take the point p_2 in the other half plane, and vice versa. By the same process, we can find $g_\infty(t)$ and its asymptotic line l_3. By repeating this process, we can find sequence of lines $\{l_i\}$ and corresponding γ_∞ such that l_i is converge to l_1. Hence we obtain the following proposition.

PROPOSITION 5. *There is an one to one correspondence between the sets of g_1 geodesics and g_0 geodesics.*

We will show the weak parallelism in the sense of [1]. This means that there is a constant $\lambda \geq 1$ such that for every point $q \in R^2$ and every geodesic γ in (M, g_1), there is a geodesic β with $\beta(0) = q$ and

$$\mathrm{dist}(\beta(t'), \gamma) \leq \lambda \mathrm{dist}(\beta(t), \gamma) \text{ for all } t, t'.$$

Then by [1] we conclude that (R^2, g_1) is the Euclidean plane. The following Lemma is useful to prove the theorem.

COMPARISON LEMMA. *Consider two scalar Jacobi differential equations*

$$y'' + K_i(x)y = 0 \quad i = 1, 2.$$

If $K_1(x) \leq K_2(x)$ for $x > 0$ and positive solutions y_i, $i = 1, 2$, go to 1 as x goes to infinity, then $y_1 \geq y_2$ for $x > 0$.

Proof. Since $y_1''y_2 - y_1y_2'' \geq 0$, by taking the integration from x to infinity we can obtain $y_1 \geq y_2$. □

RIGIDITY THEOREM. *If (R^2, g_1) is a surface without conjugate points with an asymptotic Euclidean metric and the quadratic decaying of curvature, then (R^2, g_1) is isometric to the Euclidean plane.*

Proof. Our proof consists of two steps; infinitesimal and global cases.

First we will show that for any g_1 geodesic γ, there exists a number $\tilde{\lambda} > 1$ such that

$$\frac{\sup\, y_+(\gamma(t))}{\inf\, y_+(\gamma(t'))} < \tilde{\lambda}, \quad (-\infty < t, t' < \infty),$$

where y_+ means a stable solution of the Jacobi equation. We can assume

$$\lim_{t \to \pm\infty} y_+ = 1$$

since every stable solution has the form $y_+ = C_1(1+f)$.

Consider the equation

$$y'' + K(x)y = 0,$$

where $|K(x)| \leq r(x)^{-2-\alpha}$. When $K(x) = \pm r(x)^{-2-\alpha}$, the solutions are respectively

$$z_1(x) = C_1\sqrt{x}\ \mathrm{BesselJ}(1/\alpha, \alpha/2x^{\alpha/2}) + C_2\sqrt{x}\ \mathrm{BesselY}(1/\alpha, \alpha/2x^{\alpha/2}),$$
$$z_2(x) = D_1\sqrt{x}\ \mathrm{BesselI}(1/\alpha, \alpha/2x^{\alpha/2}) + D_2\sqrt{x}\ \mathrm{BesselK}(1/\alpha, \alpha/2x^{\alpha/2}),$$

where BesselJ and BessselY denote the first and second kind of Bessel functions, BesselI and BesselK denote the first and second kind of modified Bessel functions. When $C_2 = 0$, $D_2 = 0$, z_1 and z_2 go to some positive constants as x goes to infinity. Let

$$r_0 = \text{the first zero of } \mathrm{BesselJ}(1/\alpha, x),$$

then

$$\alpha/2r_0^{-\alpha/2} = \text{the last zero of } \mathrm{BesselJ}(1/\alpha, \alpha/2x^{-\alpha/2}).$$

If $|x| > \alpha r_0^{-\alpha/2}$, then by the Comparison Lemma, we can find a constant λ. Also inside of compact set, there exist positive maximum and minimum values of y_+ for any geodesic through the compact set. Hence we prove for infinitesimal case. Now assume that there is sequences of geodesics $\{\gamma_i\}, \{\beta_i\}$ such that for all t, t'

$$\frac{\sup \operatorname{dist}(\beta_i(t'), \gamma_i)}{\inf \operatorname{dist}(\beta_i(t), \gamma_i)} = \lambda_i \to \infty$$

as i goes to ∞. The asymptotic line l_{γ_i} of γ_i is parallel to the asymptotic line l_{β_i} of β_i by the asymptotic lens equivalence.

If for any i γ_i and β_i stay in some compact set K, they have limit lines l_γ, l_β. Hence there are limiting geodesics γ, β by the one to one correspondence of g_1 and g_0 geodesics.

1-i) $\operatorname{dist}(l_\gamma, l_\beta) = c > 0$. Take a ball B with center at p such that $2r(x)^{-1} < c$ for $x \in \partial B$ and B contains K. In the interior of B, λ_i is bounded above by the compactness of B. In the exterior of B, by Proposition 1 and Lemma 2

$$\lambda_i \leq \frac{c + \frac{c}{2} + \epsilon}{c - \frac{c}{2} - \epsilon}$$

for a sufficiently large i. Hence λ_i is bounded.

1-ii) $\operatorname{dist}(l_\gamma, l_\beta) = 0$. In this case $\gamma = \beta$. Hence β_i is a geodesic variation of γ. But by the above infinitesimal argument, this cannot happen either.

If l_{γ_i} or l_{β_i} is divergent, then $\sup \operatorname{dist}(\gamma_i(t), l_{\gamma_i})$ goes to 0, and the same is true for β_i. We also will show that the following cases are also impossible.

2-i) If the distance of l_{γ_i} and l_{β_i} is greater than some positive number for any $i > i_0$, by a similar argument as 1-i) we can show that λ_i is bounded above.

2-ii) If not, for some large k we can find λ_k such that $\lambda_k > \tilde{\lambda}$, where $\tilde{\lambda}$ is a uniform constant for stable Jacobi fields.

Let J(t) be a stable Jacobi field along γ_k. Consider the geodesic variation $h(s,t) : (-\epsilon, \epsilon) \times [-a, a] \to M$ of γ_k such that

$$h(s,t) = exp_{\sigma(s)} tN(s),$$

where $\sigma(s)$ is the geodesic in $\gamma_k(-a)$ with velocity $J(-a)$ and $N(s)$ is a normal vector in $\sigma(t)$. Then by the comparison lemma and the infinitesimal argument,

$$\frac{\partial h(s,t)}{\partial s} \geq |J(s,t)| \geq A > 0,$$

where A is some positive constant independent of k. Hence the minimum distance of γ_k and $h(\epsilon/2, t)$ is bounded by a constant. Also we can take a constant ϵ for any $i > k$ by the existence theorem of ordinary differential equation applied to the geodesic equation because our Christoffel symbol is decreasing as lemma 1. Since $|J(t)|$ goes to 1 letting t to infinity, we can take any bigger a in the geodesic variation. Hence for any $i > k$ the minimum distance of γ_i and $h(\epsilon/2, t)$ is bounded by some constant which is independent of i. However l_{β_i} and l_{γ_i} get close more as i goes to infinity, for large i λ_i must be bounded by $\tilde{\lambda}$. This is a contradiction. □

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