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A CHARACTERIZATION OF SPACE FORMS

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ABSTRACT. For a Riemannian manifold (M^n, g), we consider the space $V(M^n, g)$of all smooth functions on M^n whose Hessian is proportional to the metric tensor g. It is well-known that if M^n is a space form then $V(M^n)$ is of dimension $n+2$. In this paper, conversely, we prove that if $V(M^n)$ is of dimension $\geq\ n+1$, then M^n is a Riemannian space form.

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold $(n \geq 2)$ with the Riemannian connection ∇. For $f \in C^\infty(M^n)$, the Hessian H^f of f is a symmetric (0,2) tensor field on M^n defined by

$$H^f(X, Y) = g(\nabla_X \nabla f, Y), \quad X, Y \in TM, \tag{1.1}$$

where ∇f denotes the gradient vector field of f. Let $V(M^n)$ be the space of all smooth functions on M^n whose Hessian is proportional to the metric tensor g and $m(V^n)$ denote the dimension of $V(M^n)$. Then clearly we have $m(M^n) \geq 1$ for any (M^n, g). For the Riemannian space forms $M^n = S^n(r), H^n(r)$ or E^n, we have $m(M^n) = n+2$, respectively (see §2).

Hence it is natural to ask the following question ([5,10,14]):

To what extent does $m(M^n)$ determine the geometrical and topological structure of (M^n, g) ?

And they proved, in our terminology, the following ([10,14]):

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THEOREM. *Let (M^n, g) be a complete connected Riemannian manifold with $m(M^n) \geq 2$. Then the number N of critical points of a non constant $f \in V(M^n)$ is less than or equal to 2 and M^n is conformally diffeomorphic to*

(i) *the Euclidean sphere $S^n (N = 2)$,*

(ii) *the Euclidean space E^n, or the hyperbolic space $H^n (N = 1)$,*

(iii) *the Riemannian product $I \times F$, where (F, g_F) is a complete (n-1)-dimensional Riemannian manifold and I is an open interval $(N = 0)$.*

In this article, we study the manifolds M^n with $m(M^n) \geq 3$. As a result, we prove the following:

THEOREM A. *Let (M^n, g) be a compact connected Riemannian manifold. If $m(M^n) \geq 3$, then M^n is isometric to the Euclidean sphere $S^n(r)$.*

THEOREM B. *Let (M^n, g) be a complete noncompact connected manifold. If $m(M^n) \geq n + 1$, then M^n is isometric to the Euclidean space E^n or the hyperbolic space $H^n(r)$.*

These results are sharp in the sense that (1) the ellipsoid of revolution M^n in R^{n+1} defined by $a^2x_1^2 + b^2(x_2^2 + \cdots + x_{n+1}^2) = 1, \ a \neq b,$ has $m(M^n) = 2$, (2) if M^n is the cylinder $R^{n-1} \times S^1$, then we have $m(M^n) = n$.

It is obvious that for 1-dimensional manifold M^1, we have $m(M^1) = \infty$.

2. Examples

EXAMPLE 1. Euclidean space E^n

It is straightforward to show that

$$V(E^n) = \{a \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} a_i x_i + b | a, a_1, \cdots, a_n, b \in R\},$$

where $(x_1, \cdots, x_n)$ is the rectangular coordinates for E^n.

EXAMPLE 2. Euclidean sphere $S^n(r)$

For the Euclidean sphere $S^n(r) = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} | \sum_{i=1}^{n+1} x_i^2 = r^2\}$, it is not hard to show that the restriction f of each function of the form: $\sum_{i=1}^{n+1} a_i x_i + b,\ a_i, b \in R$, belongs to $V(S^n(r))$. Conversely, each nonconstant function $f \in V(S^n(r))$ satisfies $\Delta f + (n/r^2)f = c$ (constant) (see Lemma 3.1). Hence $h = f - cr^2/n$ is a nonconstant eigenfunction of $S^n(r)$ with eigenvalue n/r^2. Therefore we see that h is a first eigenfunction of $S^n(r)$([6]), that is, f is the restriction of one of the above functions.

EXAMPLE 3. Warped product space $I \times_w F^{n-1}$ with dim I=1.

For an (n-1)-dimensional Riemannian manifold (F, g_F) let M^n be the warped product space $I \times_w F$ with metric $g = dt^2 + w(t)^2 g_F$, where I is an open interval and $w(t)$ is a positive function on I ([4,13]). Then for the function f defined by $f(t) = a\int_{t_0}^t w(t)dt + b, \quad a, b \in R$, it can be shown that H^f is proportional to g. Hence we have $m(M^n) \geq 2$.

EXAMPLE 4. Hyperbolic space $H^n(r)$

Let R_1^{n+1} be the Lorentz-Minkowski space with metric tensor $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$. Then we have

$$H^n(r) = \{x \in R_1^{n+1} | \langle x, x\rangle = -r^2, \quad x_{n+1} > 0\}.$$

It is straightforward, as in Example 2, to show that the restriction f of each function of the form : $\sum_{i=1}^{n+1} a_i x_i + b, \quad a_i, b \in R$, belongs to $V(H^n(r))$.

Conversely, using the geodesic polar coordinates centered at $(0, \cdots, 0, r)$ or equivalently, the warped product structure $H^n(r) = [0, \infty) \times_w S^{n-1}(1)$ with $w(t) = r \sinh \frac{t}{r}$, it is not hard to show that each function $f \in V(H^n(r))$ is the restriction of one of the above functions.

3. Basic formulas

Let (M^n, g) be a Riemannian manifold of n-dimension, ∇ the Riemannian connection and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

the curvature tensor. We write also $\langle X, Y\rangle$ instead of $g(X,Y)$ if this is convenient.

Note that for $f \in C^\infty(M^n)$, f belongs to $V(M^n)$ if and only if

$$\nabla_X \nabla f = \varphi X, \quad X \in TM, \quad \text{where} \quad \varphi = \Delta f/n. \tag{3.1}$$

Then it is easy to prove the following:

LEMMA 3.1. *If $f \in V(M^n)$, then we have for $X, Y \in TM$*
(i) $R(X,Y)\nabla f = X(\varphi)Y - Y(\varphi)X$
(ii) $Ric(X, \nabla f) = -(n-1)\langle X, \nabla\varphi\rangle$, *where* $\varphi = \Delta f/n$.

Thus for an Einstein manifold M^n with $Ric = (n-1)Kg$ we see that $\Delta f + nKf$ is constant.

LEMMA 3.2. *Let U be an open set without critical points of a function $f \in V(M^n)$ and let e_1 be the unit vector field in the direction of ∇f. Then we have the following:*
(i) the integral curve $\gamma(t)$ of e_1 is a geodesic and the level hypersurfaces of f are totally umbilic.
(ii) $w = |\nabla f|$ and $n\varphi = \Delta f$ are constant along the levels of f.
(iii) For any vector field X, the sectional curvature $K = K(X \wedge \nabla f)$ of the section spanned by X and ∇f is constant along the levels of f and does not depend on X.
(iv) Along $\gamma(t)$, $w(t) = w(\gamma(t))$ and $K(t) = K(\gamma(t))$ are related by

$$w''(t) + K(t)w(t) = 0. \tag{3.2}$$

Proof. The equation (3.1) reads as follows:

$$X(w)e_1 + w\nabla_X e_1 = \varphi X \quad \text{for all} \quad X \in TM,$$

or equivalently

$$e_1(w) = \varphi, \quad \nabla_{e_1} e_1 = 0 \quad \text{and} \tag{3.3}$$

$$\nabla_X e_1 = \frac{\varphi}{w} X, \quad X(w) = 0 \quad \text{for all} \quad X \perp e_1. \tag{3.4}$$

Since we have
(3.5)
$$X\varphi = X(e_1 w) = e_1(Xw) + [X, e_1](w) = (\nabla_X e_1)(w) - (\nabla_{e_1} X)(w),$$

(3.3) and (3.4) imply that $X\varphi = 0$ for all $X \perp e_1$. For any unit vector X such that $X \perp e_1$, we have from Lemma 3.1

$$\begin{aligned} K(X \wedge e_1) &= \langle R(X, e_1)e_1, X\rangle \\ &= \frac{1}{w}[\langle X(\varphi)e_1 - e_1(\varphi)X, X\rangle] \\ &= -\frac{e_1(\varphi)}{w}, \end{aligned}$$

and as in (3.5) we may prove that $X(e_1\varphi) = 0$ for all $X \perp e_1$. This completes the proof. □

For a fixed point $p \in U$, choose an orthonormal basis $e_2, \cdots, e_n$ of the level hypersurface F through p. We denote the parallel translates along $\gamma(t)$ of $e_2, \cdots, e_n$ by the same notation. For each $i \in \{2, \cdots, n\}$ let $\eta_i(s)$ be the integral curve of e_i in F with $\eta_i(0) = p$, and $x(s,t)$ be the one parameter family $\exp_{\eta_i(s)}(te_1)$ of geodesics. Then $J_i(t) = x_s(0,t)$ is a Jacobi vector field along γ, so that it satisfies the Jacobi equation

$$J_i''(t) + R(J_i, e_1)e_1 = 0 \quad \text{with} \quad J_i(0) = e_i, \quad J_i'(0) = \frac{w'(0)}{w(0)}e_i, \tag{3.6}$$

where $w'(t)$ denotes the derivative with respect to t.

Since e_i is parallel along γ, (3.2) and the Jacobi equation show that $\langle J_i(t), e_1\rangle = 0$ and

$$\langle J_i(t), e_j\rangle'' + K(t)\langle J_i(t), e_j\rangle = 0, \quad j \in \{2, \cdots, n\}. \tag{3.7}$$

Hence from (3.2) and the initial condition we obtain

$$J_i(t) = \frac{w(t)}{w(0)}e_i. \tag{3.8}$$

This shows that the metric g of M is locally given by $g = dt^2 + \frac{w(t)^2}{w(0)^2}g_F$, that is, M is locally the warped product space $I \times_{w(t)/w(0)} F$.

Thus from Example 3 of §2, we obtain the following ([10,14]):

LEMMA 3.3. *The following conditions are equivalent:*

(i) *There exists a neighborhood U of p and a function $f \in V(U)$ with $\nabla f(p) \neq 0$.*

(ii) *There exists a neighborhood U of p such that U is a warped product space $I \times_w F$ with 1-dimensional base I.*

In his Thesis ([7]), K. L. Easley found an equivalent condition of the above lemma (see also [1]).

The situation is quite different near a critical point of f. W. Kühnel proved that the critical points of a nonconstant function $f \in V(M^n)$ are isolated ([10, 14]). Around a critical point $p \in M$ of a nonconstant function $f \in V(M)$ we may prove the following ([10]):

LEMMA 3.4. *The following conditions are equivalent:*

(i) *There exists a neighborhood U of p and a nonconstant function $f \in V(U)$ with $\nabla f(p) = 0$.*

(ii) *There exists polar coordinates $(t, u_1, \cdots, u_{n-1})$ in a neighborhood of p and an even function $f = f(t)$ with $f'(0) = 0$ and $f''(0) \neq 0$ such that*

$$g = dt^2 + \frac{f'(t)^2}{f''(0)^2} g_1,$$

where g_1 denotes the metric of the Euclidean unit sphere $S^{n-1}(1)$.

(iii) *There exists a function $\lambda(t)$ defined on $(0, t_0)$ such that, for each $\xi \in T_pM$, $|\xi| = 1$, and each $t \in (0, t_0)$ the shape operator $S_p(m)$ of the geodesic sphere $G(p, t)$ at $m = \exp_p(t\xi)$ satisfies $S_p(m) = \lambda(t)I$.*

Proof. Equivalence of (i) and (ii) was given in [10]. (i) $\Rightarrow$ (iii) is given by (3.4) with $\lambda(t) = w'(t)/w(t)$. Suppose that (iii) holds. Then Theorem 12 in [9] implies that, with respect to any normal coordinates centered at p, the metric tensor g is given by the formula:

$$g = h^2(t) \sum_{i=1}^{n} (dx^i)^2 + \frac{1 - h^2(t)}{t^2} (\sum_{i=1}^{n} x^i dx^i)^2,$$

where $h(t) = \exp\{\int_0^t (\lambda(r) - \frac{1}{r}) dr\}$, and $t^2 = \sum_{i=1}^n x_i^2$.

Since $\sum_{i=1}^n (dx^i)^2 = dt^2 + t^2 ds_1^2$, where ds_1^2 is the line element of the unit sphere in T_pM, we get by substitution

$$g = dt^2 + w^2(t) ds_1^2, \qquad w(t) = th(t).$$

Now we may prove that $f(t) = \int_0^t w(r)dr$ satisfies (3.1) on the geodesic ball of radius t_0 around p. □

Note that for a critical point p of $f \in V(M^n)$ the level hypersurfaces of f coincide with the geodesic spheres around p and p is an isotropic point with curvature $K(0)$. For the space form $E^n, S^n(r)$ or $H^n(r)$ we have

$$\lambda(t) = \frac{1}{t},\ \frac{1}{r}\cot\frac{t}{r} \quad \text{or} \quad \frac{1}{r}\coth\frac{t}{r},$$

respectively.

4. Proofs

In this section we prove the main theorems and we assume that M^n is connected and complete.

First, we give some lemmas. Note that for any nonconstant function $f \in V(M^n)$, the number N of critical points of f is less than or equal to 2 and M^n is conformally diffeomorphic to $S^n(N=2)$, E^n or $H^n(N=1)$ and $I \times F(N=0)$ ([10]).

LEMMA 4.1. *Let $f_1, f_2 \in V(M^n)$ with $\nabla f_1(p) = \nabla f_2(p) = 0$. Then $\{f_1, f_2, 1\}$ is linearly dependent on M^n.*

Proof. We may assume that f_1, f_2 are nonconstant functions. For any radial geodesic $\gamma(t)$ emanating from p, let $w_i(t) = |\nabla f_i|(\gamma(t)),\ i = 1, 2$. Then $w_i(t)$ satisfies

$$(3.2) \qquad w_i''(t) + K(t)w_i(t) = 0 \quad \text{with} \quad w_1(0) = w_2(0) = 0.$$

Hence we have $w_2(t) = aw_1(t)$, where $a = w_2'(0)/w_1'(0)$. Since $f_i(t) = \int_0^t w_i(t)dt + f_i(0)$, this completes the proof. □

LEMMA 4.2. *Let $f_1, f_2, \cdots, f_k \in V(M^n)$ with $k \le n$. If $\dim\langle\{\nabla f_1(p), \cdots, \nabla f_k(p)\}\rangle \le k-1$ at each point p in an open set U, then $\{f_1, \cdots, f_k, 1\}$ is linearly dependent on M^n.*

Proof. For $k = 1$, suppose that $\nabla f_1 \equiv 0$ on an open set U. Then f_1 must be constant, because f_1 has infinitely many critical points.

Now assume that the lemma holds for $k \leq n-1$. Suppose that $\dim\langle\{\nabla f_1(p), \cdots, \nabla f_{k+1}(p)\}\rangle \leq k$ on U with $k+1 \leq n$. Then by induction hypothesis we may assume that $U_1 = \{p \in U | \dim\langle\{\nabla f_1(p), \cdots, \nabla f_k(p)\}\rangle = k\}$ is a nonempty open set. On U_1 we have $\nabla f_{k+1} = h_1 \nabla f_1 + \cdots + h_k \nabla f_k$. And (3.1) shows that for all X we have

$$(4.1)\quad \varphi_{k+1} X = X(h_1)\nabla f_1 + \cdots + X(h_k)\nabla f_k + (h_1\varphi_1 + \cdots + h_k\varphi_k)X,$$

where $\varphi_i = \Delta f_i / n, i = 1, \cdots, k+1$. Since $k \leq n-1$ we can choose X so that X is orthogonal to $\{\nabla f_1, \cdots, \nabla f_k\}$. Hence (4.1) implies that

$$(4.2)\qquad \varphi_{k+1} = h_1\varphi_1 + \cdots + h_k\varphi_k.$$

By (4.2) together with (4.1) we see that $h_1, \cdots, h_k$ are constants $a_1, \cdots, a_k$, respectively. Therefore the gradient of $f_{k+1} - (a_1 f_1 + \cdots + a_k f_k) \in V(M^n)$ vanishes on U_1, in particular, it has infinitely many critical points. Thus it must be constant on M^n. This completes the proof. □

LEMMA 4.3. *Let c be a regular value of $f_1 \in V(M^n)$ and F_1 be the hypersurface $f_1^{-1}(c)$. Then*

(i) *For any $f \in V(M^n)$, the restriction $\tilde{f} = f_{|F_1}$ of f belongs to $V(F_1)$.*

(ii) *If $\{f_1, \cdots, f_k, 1\}$ is a linearly independent subset of $V(M^n)$, then $\{\tilde{f}_2, \cdots, \tilde{f}_k, 1\}$ is linearly independent. Hence we have $m(F_1) \geq m(M^n) - 1$.*

Proof. (i) Let $\tilde{\nabla}$ be the induced connection of F_1 and $\tilde{\nabla}\tilde{f}$ be the gradient vector field of $\tilde{f}$ on F_1. Then, using Lemma 3.2, a direct computation shows that for all $X \in TF_1$

$$\tilde{\nabla}_X \tilde{\nabla}\tilde{f} = \left(\varphi - \frac{\varphi_1\langle\nabla f, \nabla f_1\rangle}{\langle\nabla f_1, \nabla f_1\rangle}\right)X,$$

where $\varphi = \Delta f / n$ and $\varphi_1 = \Delta f_1 / n$.

(ii) Suppose that $a_2\tilde{f}_2 + \cdots + a_k\tilde{f}_k + b = 0$ on F_1. Then we have on F_1, $\nabla(a_2f_2+\cdots+a_kf_k) = h\nabla f_1$ for some function h on F_1. By (3.1) we get for all $X \in TF_1$, $(a_2\varphi_2 + \cdots + a_k\varphi_k)X = X(h)\nabla f_1 + h\varphi_1 X$. Thus we see that h is a constant c, which implies that $cf_1 - (a_2f_2 + \cdots + a_kf_k)$ is a function in $V(M^n)$ which has infinitely many critical points. Therefore $cf_1-(a_2f_2+\cdots+a_kf_k)$ must be constant. This completes the proof. □

Proof of Theorem A. Suppose that $\{f_1, f_2, 1\}$ is a linearly independent subset of $V(M^n)$. Since M^n is compact, M^n is conformally diffeomorphic to S^n([10]) and there exist exactly two critical points p_1, q_1 of f_1. Let p_2 be a critical point of f_2 and let $d = d(p_1, p_2)$ and $l = d(p_1, q_1)$. Then from Lemma 4.1 we have $0 < d < l$. Consider the geodesic $\gamma(t)$ with $\gamma(0) = p_1$, $\gamma(d) = p_2$, then we have $\gamma(l) = q_1$([10]).

Note that the geodesic sphere $G(p_2, d)$ passes through p_1 and meets every geodesic sphere $G(p_1, t)$, $0 < t \le d$. For a fixed $t_0 \in (0, d]$ let q be a point in $G(p_2, d) \cap G(p_1, t_0)$ and let η_1, η_2 be the geodesic from p_1, p_2 through q, respectively. Then we have $\eta_1(t_0) = \eta_2(d) = q$ and $\{\dot\eta_1(t_0), \dot\eta_2(d)\}$ is linearly independent. Thus Lemma 3.2 implies that $K_1(t_0) = K_1(\dot\eta_2(d) \wedge \dot\eta_1(t_0)) = K_2(d)$, where $K_1(t)$ and $K_2(t)$ are the sectional curvature functions corresponding to f_1 and f_2, respectively. Since $G(p_1, d) = G(q_1, l - d)$, for a fixed $t_0 \in [d, l)$, we may as above prove that $K_1(t_0) = K_2(l - d)$. Hence $K_1(t)$ is a constant k on $(0, l)$ hence on $[0, l]$.

Note that $w_1(t)$ satisfies

$$(3.2) \qquad w_1''(t) + kw_1(t) = 0 \quad \text{with} \quad w_1(0) = w_1(l) = 0.$$

Thus k must be positive (say, $1/r^2$), so that we have $w_1(t) = a\sin(t/r)$, $a \in R$. The shape operator $S_{p_1}(\gamma(t))$ of the geodesic sphere $G(p_1, t)$ at $\gamma(t)$ satisfies $S_{p_1}(\gamma(t)) = w_1'(t)/w_1(t) = (1/r)\cot(t/r)$, which implies that the sectional curvature K_M of M^n is $k = 1/r^2$([9]). Since M^n is simply connected, M^n is isometric to the Euclidean sphere $S^n(r)$. □

Proof of Theorem B. Let $\{f_1, \cdots, f_n, 1\}$ be a linearly independent subset of $V(M^n)$. Then Lemma 4.2 implies that there exists an open dense subset U such that $\{\nabla f_1(p), \cdots, \nabla f_n(p)\}$ is linearly independent for all $p \in U$. By Lemma 3.1 we see that the sectional curvature of M^n is a constant K on U hence on M^n. Since M^n is noncompact, K is nonpositive. If there is a nonconstant function f in $V(M^n)$ such that $\nabla f(p) = 0$ for some $p \in M$, then M^n must be simply connected, which implies that M^n is isometric to R^n or $H^n(r)$. Thus the proof is completed.

Now suppose that $\nabla f(p) \neq 0$ for all $p \in M$ and for all nonconstant $f \in V(M^n)$. Then by Lemma 4.3, we see that M^n is isometric to the warped product space $R \times_{w_1} F_1$, where F_1 is a level hypersurface of f_1. Note that F_1 is also complete and connected. And Lemma 4.3 shows that $m(F_1) \geq n$ and F_1 is also a warped product space $R \times_{w_2} F_2$, where w_2 is the length function of gradient of the restriction $\tilde{f}_2$ of f_2 on F_1 and F_2 is a level hypersurface of $\tilde{f}_2$ in F_1.
Inductively, we have the following:

$$\begin{aligned} M^n &= R \times_{w_1} F_1 \\ &= R \times_{w_1} (R \times_{w_2} F_2) \\ &= R \times_{w_1} (R \times_{w_2} (R \times_{w_3} \cdots (R \times_{w_{n-1}} F_{n-1}) \cdots). \end{aligned}$$

Note that each $F_k = R \times_{w_{k+1}} F_{k+1}$ is a complete connected $(n-k)$-dimensional manifold with $m(F_k) \geq n - k + 1$ and $R \times_{w_{n-1}} F_{n-1}$ is a warped product space with $m(R \times_{w_{n-1}} F_{n-1}) \geq 3$. Since we can prove that $V(R \times_{w_{n-1}} S^1) = \{a \int_0^t w_{n-1}(s)ds + b | a, b \in R\}$, F_{n-1} must be the Euclidean line. This implies that M^n is isometric to $R \times_{w_1} R \times \cdots \times_{w_{n-1}} R$, in particular, M^n is simply connected. This completes the proof. □

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