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ON THE LAWS OF NILPOTENT POINTED-GROUPS

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Abstract. A pointed-group is an ordered pair (G, c) where G is a group and c is a specific element of G. Thus a pointed-group is a group together with a distinguish element. The aim of this paper is to generalize the result proved by R. C. Lyndon in [4], that every nilpotent group variety is finitely based for its laws.

1. Introduction

In [4] R. C. Lyndon proved that every nilpotent group variety has a finite basis for its laws. (That is, there is a finite set of laws of which every law is a consequence). Here we shall examine the analogous statement for nilpotent pointed-groups, where a pointed-group is a pair (G, g) consisting of a group G together with a distinguished element g of G. (The idea of pointed-groups comes from the category of pointed-sets), see for example [6].

By a law of a pointed-group (G, g) we shall mean a word w of the free group on the countable set $\{y, x_1, x_2, \cdots, \cdots\}$ such that w always becomes equal to the identity element of G when g is substituted for y and arbitrary elements of G are substituted for $x_1, x_2, \cdots$ (For example, $[y, x_1]$ is law of (G, g) if g is central in G or G is abelian). Included among the laws of (G, g) are the laws of the group G, or more precisely, those words in the variable $x_1, x_2, \cdots$, which are laws of G, thus the idea of laws of a pointed-group generalizes the idea of laws of a group and the purpose of this paper is to generalize the result proved by R. C. Lyndon in [4], that every variety of nilpotent groups is finitely based. Detailed information concerning varieties of groups may be found in [7].

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A pointed-group may be regarded as a group with an extra nullary operation and therefore it is an algebra in the sense of a universal algebra. The more general concepts related to universal algebras and varieties of algebras are described in [2]. It is useful to note the form of some of these concepts take for pointed-groups.

As mentioned above, the factor algebra of (G, g) is $(G/N, gN)$ where N is a normal sub-group of G not necessarily containing g. The Cartesian product of the family $\{(G_\lambda, g_\lambda) | \lambda \in \Lambda\}$ is (G, g) where G is the Cartesian product of $\{G_\lambda | \lambda \in \Lambda\}$ and g is the element of G with value g_λ at λ for all $\lambda \in \Lambda$. A generating set for a pointed-group (G, g) is a subset S of G such that $S \cup \{g\}$ generates G. If (G, g) can be generated by a set with n or fewer elements, then we shall say that (G, g) is an n-generator pointed-group. By an endomorphism of (G, g) we mean a homomorphism α from (G, g) to (G, g) such that $g\alpha = g$. Moreover, if (G, g) is a pointed-group and N is a normal sub-group of G, then we say that N is a normal admissible sub-group of (G, g) if $N\alpha \leq N$ for every endomorphism α of (G, g).

A variety of pointed-groups is the class of all pointed-groups in which the elements of some given set of words are all laws. Equivalently, it is a class closed under the operations of taking factor algebras, sub-algebras, Cartesian products and algebras isomorphic to these. We shall write $\mathcal{V}$ to denote the variety of pointed-groups.

2. Notations and definitions

We call a pointed-group (G, g) nilpotent of class c if G is a nilpotent group of class c. Here we shall discuss varieties of pointed-groups in which every pointed-group is nilpotent. Our aim is to generalize the result proved by R. C. Lyndon in [4], that every variety of nilpotent groups is finitely based.

We shall need some notational definitions in the free pointed-group (X, y) generated by $x_1, x_2, \cdots$.

$$\text{Put } X_{(1)} = X$$

and define for $c \geq 1$ $X_{(c+1)} = [X_{(c)}, X]$, i.e., the sub-group of X generated by the elements of the form $[a, b]$ where $a \in X_{(c)}$ and $b \in X$. It is easy to see that each $X_{(c)}$ is a normal admissible sub-group of (X, y). Also, note that $X_{(c)} = [x_1, x_2, \cdots, x_c](X, y)$ for $c \geq 2$, for example, see Theorem 10.2.1 in [3].

Now for each positive integer c, we write $\mathcal{N}_c$ to denote the variety of nilpotent pointed-groups defined by the word $[x_1, x_2, \cdots, x_{(c+1)}]$ whose closure is $X_{(c+1)}$. Thus $\mathcal{N}_0$ is the variety of all trivial pointed-groups and $\mathcal{N}_1$ is the variety of all abelian pointed-groups (we call a pointed-group (G, g) abelian if G is an abelian group).

More generally, $\mathcal{N}_c$ is the variety of all nilpotent pointed-groups which are nilpotent of class at most c.

A variety $\mathcal{V}$ of pointed-groups is called *nilpotent* if $\mathcal{V} \subseteq \mathcal{N}_c$ for some c. We say that $\mathcal{V}$ has a class c if $\mathcal{V} \subseteq \mathcal{N}_c$ and $\mathcal{V} \nsubseteq \mathcal{N}_{c+1}$.

We will prove the following theorem which is a generalization of the result in [4].

THEOREM 2.1. *The laws of any variety of nilpotent pointed-groups are finitely-based.*

Now in order to prove the above Theorem 2.1, we need to establish some results.

Now for each positive integer n, δ_n denotes the endomorphism of (X, y) determined by $x_n\delta_n = 1$ and $x_i\delta_n = x_i$, for $i \neq n$. Thus the effect of δ_n on a word w , is to delete all occurrences of x_n from w

Now for all positive integers i, m and n, it is easy to check that $x_i\delta_m\delta_n = x_i\delta_n\delta_m$ and $x_i\delta_n^2 = x_i\delta_n$. Thus by Corollary 1.4.6 in [1], we have the relations, $\delta_n\delta_m = \delta_m\delta_n$ and $\delta_n^2 = \delta_n$.

Moreover, if w is any word we write $w(1 - \delta_n)$ to denote $w(w\delta_n)^{-1}$. Thus we have $w = w(1 - \delta_n)w\delta_n$.

Also, we have

$$\begin{aligned} w(1 - \delta_m)\delta_n &= w(w\delta_m)^{-1}\delta_n \\ &= w\delta_n(w\delta_m\delta_n)^{-1} \\ &= w\delta_n(w\delta_n\delta_m)^{-1} \\ &= w\delta_n(1 - \delta_m) \end{aligned}$$

and

$$\begin{aligned} w(1-\delta_n)\delta_n &= w(w\delta_n)^{-1}\delta_n \\ &= (w\delta_n)(w\delta_n\delta_n)^{-1} \\ &= (w\delta_n)(w\delta_n{}^2)^{-1} \\ &= (w\delta_n)(w\delta_n)^{-1} \\ &= 1 \end{aligned}$$

Hence we have also the relations $w(1-\delta_n)\delta_n = 1$ and $w(1-\delta_m)\delta_n = w\delta_n(1-\delta_m)$. Note that $w(1-\delta_n)$ and $w\delta_n$ are consequences of w. Thus we have:

THEOREM 2.2. *Let T be a finite set of positive integers. Let w be a word in (X, y) such that $w\delta_t = 1$ for all $t \in T$. then $w \in X_{(n)}$ where $\mid T \mid = n$.*

Proof. See Corollary 33.38 of [7]. □

THEOREM 2.3. *Let w be an element of $X_{(n)}$ than there are words w_T for each subset T of $\{1, 2, 3, \cdots, n\}$ such that*

(i) *w is a product of the words w_T in some order.*
(ii) *Each w_T is a consequence of w.*
(iii) *Each $w_T\delta_t = 1$ for all $t \in T$ and also w_T is a word only in the variables y and $x_t \in X$ where $t \in T$.*

Proof. Assume that w is a word in the variables y and $x_1, x_2, \cdots, x_n$. Then consider,

(1) $$w = w(1-\delta_1)w\delta_1$$

But,

$$w(1-\delta_1) = w(1-\delta_1)(1-\delta_2)w(1-\delta_1)\delta_2$$

and

$$w\delta_1 = w\delta_1(1-\delta_2)w\delta_1\delta_2$$

Thus (1) gives

$$w = w(1-\delta_1)(1-\delta_2)w(1-\delta_1)\delta_2 w\delta_1(1-\delta_2)w\delta_1\delta_2,$$

i.e.,

$$w = w(1-\delta_1)(1-\delta_2)w(1-\delta_1)\delta_2 w(1-\delta_2)\delta_1 w\delta_1\delta_2.$$

Continuing in this way we get,

$$w = \prod_{T\in\{1,2,3,\cdots,n\}} w_T$$

where the product is taken over the subsets T of $N = \{1, 2, \cdots, n\}$ and where

$$w_T = w\prod_{u\in T}(1-\delta_u)\prod_{t'\in N\setminus T}\delta_{t'}\,.$$

Now apply δ_t to w_t, $t \in T$ we have

$$w_T\delta_t = \left(w\prod_{u\in T}(1-\delta_u)\prod_{t'\in N\setminus T}\delta_{t'}\right)\delta_t\,.$$

But, δ_t and $\delta_{t'}$ commute, also δ_t and $(1-\delta_u)$commute and $w\cdots(1-\delta_t)\delta_t\cdots = 1$. Hence it follows that $w_T\delta_t = 1$. Since this holds for all $t \in T$ and also w_T is a word only in the variables y and x_t where $t \in T, w_T$ has the form

$$v\prod_{t'\in N\setminus T}\delta_{t'}$$

where,

$$v = w\prod_{u\in T}(1-\delta_u).$$

and hence does not contain $x_{t'}, t' \notin T$. Thus the conditions (i) and (iii) are satisfied. Now condition (ii) is an easy consequence of the properties of δ_n and $(1-\delta_n)$ described above. Thus the proof is complete. □

3. Proof of Theorem 2.1

Suppose $\mathcal{V}$ is a variety of nilpotent pointed-groups of class at most c, i.e., $\mathcal{V} \subseteq \mathcal{N}_c$. Let V be the normal admissible subgroup of (X, y) which defines $\mathcal{V}$. Let $w \in V$ and suppose that $w = w(y, x_1, x_2, \cdots, x_n)$. Then by Theorem 2.3, w is equivalent to a set of words $\{w_T\}$ where each T is a finite set of positive integers, with the properties given in the statement of Theorem 2.3.

Now suppose $|T| \geq c+1$, then by Theorem 2.2, $w_T \in X_{(c+1)}$. Therefore, w_T is a consequence of the word $[x_1, x_2, \cdots, x_{c+1}]$. But if $|T| < c+1$. Then (by the change of variables) w_T is equivalent to an element of $V \cap X_{(c)} = V_{(c)}$. Thus in either case, w_T is a consequence of $V_{(c+1)}$. Hence, it follows that w is a consequence of $V_{(c+1)}$. Thus V is equivalent to $V_{(c+1)}$. Now by Lemma 2.4.3 in [1] we have

$$\mathcal{V}(X_{(c+1)}, y) = \mathcal{V}(x, y) \cap X_{(c+1)},$$

i.e.,

$$\mathcal{V}(X_{(c+1)}, y) = V_{(c+1)}.$$

Now since $\mathcal{V} \subseteq \mathcal{N}_c$, so we have

$$\mathcal{N}_c(X_{(c+1)}, y) \subseteq \mathcal{V}(X_{(c+1)}, y).$$

Thus by Corollary 2.2.9 in [1], $\mathcal{N}_c(X_{(c+1)}, y) = K(say)$, is normal in $X_{(c+1)}$. Thus $K \subseteq V_{(c+1)}$ and therefore, $V_{(c+1)}/K$ is a sub-group of $X_{(c+1)}/K$. But $(X_{(c+1)}/K, yK) \in \mathcal{N}_c$ (by Corollary 2.2.10 in [1]). Therefore, $X_{(c+1)}/K$ is a nilpotent group of class at most c. But $X_{(c+1)}$ is a finitely generated nilpotent group. Therefore, $X_{(c+1)}/K$ is a finitely generated nilpotent group.

Now Theorem 9.16 of [5], states that in a finitely generated nilpotent group, every sub-group is finitely generated. Hence it follows that $V_{(c+1)}/K$, being a sub-group of $X_{(c+1)}/K$ is finitely generated. Suppose, $v_1K, v_2K, \cdots, v_mK$ are generators of $V_{(c+1)}/K$, i.e.,

$$V_{(c+1)}/K = \langle v_1, K, \cdots, v_mK \rangle$$

Then,

$$V_{(c+1)} = \langle v_1, v_2, \cdots, v_m, K \rangle$$

For, if $V_{(0)} = \langle v_1, \cdots, v_m, K \rangle$ then clearly $V_{(0)}$ is contained in $V_{(c+1)}$. Now let $v \in V_{(c+1)}$, then $vK \in V_{(c+1)}/K$. Thus vK can be written in the form

$$\begin{aligned} vK &= (v_{i_1}K)^{\epsilon_1}(v_{i_2}K)^{\epsilon_2} \cdots (v_{i_n}K)^{\epsilon_n} \\ &= (v_{i_1}^{\epsilon_1} v_{i_2}^{\epsilon_2} \cdots v_{i_n}^{\epsilon_n})K. \end{aligned}$$

Therefore, v has the form

$$v_{i_1}^{\epsilon_1} v_{i_2}^{\epsilon_2} \cdots v_{i_n}^{\epsilon_n} k,$$

where $k \in K$.

Hence $v \in V_{(0)}$, i.e., $V_{(c+1)}$ is contained in $V_{(0)}$. Thus $V_{(c+1)} = V_{(0)}$ as desired. Thus V is equivalent to $K \cup \{v_1, v_2, \cdots, v_m\}$. But, $K = \mathcal{N}_c(X_{(c+1)}, y) \leq \mathcal{N}_c(X, y)$ and $\mathcal{N}_n(X, y)$ is the set of all consequences of $[x_1, x_2, \cdots, x_{c+1}]$. Therefore, V is equivalent to the set $\{v_1, v_2, \cdots, v_m, [x_1, x_2, \cdots, x_{c+1}]\}$. Thus $\mathcal{V}$ is finitely based and the proof of Theorem 2.1 is complete.

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