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# ON THE FINSLER SPACES WITH $f$-STRUCTURE

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ABSTRACT. In this paper the properties of the Finsler metrics compatible with an $f$-structure are investigated.

## 1. Introduction

The $f$-structure in a Riemannian manifold is defined by a tensor field $f$ of (1,1) type such that $f^3 + f = 0$ [7]. The $f$-structure may be regarded as a generalization of the almost complex structure and the almost contact structure.

On the other hand, the Finsler space admitting an almost complex structure was introduced by several authors. Moreover, I. Hasegawa, K. Yamauchi and H. Shimada [2] introduced the almost contact structure on Finsler space. Recently, Y. Ichijyō [4] introduced the notion of the Finsler metrics compatible with $f$-structure which is generalization of the almost complex structure and the almost contact structure. And the first author of the present paper and H. Y. Park [6] introduced the notion of the Finsler metrics compatible with $\varphi(4,2)$-structure.

In the present paper, we treat the Finsler spaces whose metrics is compatible with $f$-structure. These Finsler spaces will be called $(f, L)$-*manifolds*. First, we find the condition for the Finsler metric on $(f, L)$-manifolds to be a Riemannian metric. Secondly, some properties of the $(f, L)$-manifolds with vanishing $h$-covariant derivative of $f$-structure are investigated.

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## 2. Preliminaries

Let $M$ be an $n$-dimensional differentiable manifold admitting a tensor field $f^i{}_j(x)$ of type (1,1) satisfying

$$f^t{}_j f^r{}_t f^i{}_r + f^i{}_j = 0. \tag{2.1}$$

If we put

$$\ell^i{}_j = -f^i{}_r f^r{}_j, \quad m^i{}_j = f^i{}_r f^r{}_j + \delta^i{}_j, \tag{2.2}$$

we have

$$\begin{aligned} &\ell^i{}_j + m^i{}_j = \delta^i{}_j, \quad f^r{}_j \ell^i{}_r = \ell^r{}_j f^i{}_r = f^i{}_j, \\ &\ell^r{}_j \ell^i{}_r = \ell^i{}_j, \quad m^r{}_j m^i{}_r = m^i{}_j, \quad f^r{}_j f^t{}_r \ell^i{}_t = -\ell^i{}_j, \\ &f^r{}_j m^i{}_r = m^r{}_j f^i{}_r = 0, \quad \ell^r{}_j m^i{}_r = m^r{}_j \ell^i{}_r = 0. \end{aligned} \tag{2.3}$$

Hence, the operators $\ell^i{}_j$ and $m^i{}_j$ applied to the tangent space $T(M)$ at each point of $M$ are complementary projection operators. Thus there exist complementary distribution $\mathfrak{L}$ and $\mathfrak{M}$ corresponding to $\ell^i{}_j$ and $m^i{}_j$ respectively. If the rank of $f^i{}_j$ is $r$, then we call such a structure *an $f$-structure of rank $r$* [7]. The manifold admitting an $f$-structure of rank $r$ is called *an $f$-manifold.* Moreover $f^i{}_j$ is an almost complex structure operator on $\mathfrak{L}$ and, at same time, $f^i{}_j$ is null operator on $\mathfrak{M}$. If the rank of $f$ is $n$, then $\ell^i{}_j = -\delta^i{}_j$ and $m^i{}_j = 0$, so that we find the $f$-structure of rank $n$ is an almost complex structure.

It is well known that, in an $f$-manifold, there exists a positive definite Riemannian metric $a_{ij}(x)$ with respect to which the distribution $\mathfrak{L}$ and $\mathfrak{M}$ are orthogonal and such that

$$a_{ij}(x) = a_{pq}(x) f^p{}_i f^q{}_j + a_{ip}(x) m^p{}_j, \qquad f_{ij} = -f_{ji}, \tag{2.4}$$

where $f_{ij} = a_{ip} f^p{}_j$.

The integrability condition of distribution $\mathfrak{L}$ is given by [7]

$$\ell^r{}_j \ell^t{}_k (\partial_r m^i{}_t - \partial_t m^i{}_r) = 0 \quad (\partial_r = \partial/\partial x^r). \tag{2.5}$$

The integral manifold of $\mathfrak{L}$ may be represented by parametric equation $x^i = x^i(u^a), a = 1, 2, \cdots, r$. If we put $B^i{}_a = \partial_a x^i \quad (\partial_a = \partial/\partial u^a)$, we can induce an almost complex structure $'f^a{}_b$ on integral manifold of $\mathfrak{L}$ by $'f^a{}_b = B_b{}^i B^a{}_j f^j{}_i$, where $B^a{}_j$ satisfies $B^a{}_i B_b{}^i = \delta^a{}_b$ and $B^a{}_i B_a{}^j = \ell^j{}_i$. Let $'N_{cb}{}^a, N_{kj}{}^i$ be the Nijenhuis tensors for the almost complex structure $'f^a{}_b$ and the $f$-structure $f^i{}_j$ respectively. It is well known that $'N_{cb}{}^a$ and $N_{kj}{}^i$ is related as follows [7]

$$'N_{cb}{}^a = B_c{}^j B_b{}^i B^a{}_k N_{ji}{}^k. \tag{2.6}$$

When the distribution $\mathfrak{L}$ is integrable and the almost complex structure induced on the integral manifold by $f$-structure is also integrable, we say that the $f$-structure is *partially integrable.*

## 3. $(f, L)$-manifold

Let us assume that a differentiable $n$-manifold $M$ admits a Finsler metric function $L(x, y)$. This Finsler metric function $L(x, y)$ satisfies

$$\begin{aligned} &L(x, ky) = kL(x, y) \quad \text{for any} \quad k > 0, \\ &g_{ij}(x, y)\xi^i\xi^j \quad \text{is positive definite,} \end{aligned} \tag{3.1}$$

where $g_{ij}(x, y) = \dot\partial_i \dot\partial_j L^2(x, y)/2 \quad (\dot\partial_i = \partial/\partial y)$.

The tangent space $T_x(M)$ at any point $x$ in $M$ can be regarded as $n$-dimensional normed linear space such that the norm $\|y\|$ of any tangent vector $y$ in $T_x(M)$ as follows

$$\|y\| = L(x, y). \tag{3.2}$$

Then it is easy to verify that $T_x(M)$ is a finite Banach space. The distribution $\mathfrak{L}$ at $x$ is considered as the tangent subspace in $T_x(M)$ and the $f$-structure can be considered as an almost complex structure on $\mathfrak{L}$. For any complex number $\tilde c = |c|(\cos\theta + i\ \sin\theta)$, we define the scalar product of $\tilde c$ and tangent vector $\ell y$ on $\mathfrak{L}$ as follows

$$\begin{aligned} \tilde c\ \ell y &= |\tilde c|(\cos\theta\ \delta^i_j + \sin\theta\ f^i{}_j)\ell^j{}_k y^k \\ &= |\tilde c|(\cos\theta\ \ell^i{}_k y^k + \sin\theta\ f^i{}_k y^k) \end{aligned}$$

for any $\theta$. If we put ${\varphi_\theta}^i{}_j = \cos\theta\ \delta^i_j + \sin\theta\ f^i{}_j$, we have from (3.2)

$$\|\tilde{c}\cdot \ell y\| = |\tilde{c}|\|\varphi_\theta \ell y\| = |\tilde{c}|L(x, \varphi_\theta \ell y).$$

Therefore, $\mathfrak{L}$ is a complex Banach space if and only if

$$L(x, \varphi_\theta \ell y) = L(x, \ell y). \tag{3.3}$$

From the homogeneity of $g_{ij}(x,y)$ in $y$ it is easy to verify that (3.3) is equivalent to

$$g_{pq}(x, \varphi_\theta \ell y){\varphi_\theta}^p{}_r{\varphi_\theta}^q{}_t\ell^r{}_i\ell^t{}_j = g_{pq}(x.\ell y)\ell^p{}_i\ell^q{}_j. \tag{3.4}$$

The manifold satisfying (3.3) or (3.4) is said to be an $(f, L)$-*manifold.*

On the other hand, M. Fukui [1] and Y. Ichijyō [3] proved that if a Finsler metric $g_{ij}(x,y)$ and an almost complex structure $F^i{}_j(x)$ satisfy the condition

$$g_{ij}(x,y) = g_{pq}(x,y)F^p{}_iF^q{}_j,$$

then $g_{ij}$ is a Riemannian.

Now we consider the result above analogously on an $(f, L)$-manifold. We suppose that an $(f, L)$-manifold satisfies

$$g_{ij}(x, \ell y) = g_{pq}(x, \ell y)f^p{}_if^q{}_j, \tag{3.5}$$

where $f^p{}_i$ is an $f$-structure.

Differentiating (3.5) with respect to $y^k$, we have

$$C_{ijr}(x, \ell y)\ell^r{}_k = C_{pqr}(x, \ell y)f^p{}_if^q{}_j\ell^r{}_k, \tag{3.6}$$

where $2C_{ijr}(x,y) = \dot{\partial}_r g_{ij}(x,y)$.
Transvecting (3.6) with $\ell^j{}_s$ and using (2.3), we have

$$C_{ijr}(x, \ell y)\ell^r{}_k\ell^j{}_s = C_{pqr}(x, \ell y)f^p{}_if^q{}_s\ell^r{}_k. \tag{3.7}$$

On the other hand, by the symmetry of $C_{ijr}$ in all indices, we get

$$C_{ijr}(x, \ell y)\ell^r{}_k\ell^j{}_s = C_{ijr}(x, \ell y)\ell^r{}_s\ell^j{}_k,$$

that is,

$$C_{pqr}(x,\ell y)f^p{}_i f^q{}_s \ell^r{}_k = C_{pqr}(x,\ell y)f^p{}_i f^q{}_k \ell^r{}_s. \tag{3.8}$$

From (3.6) and (3.8), we have

$$C_{ijr}(x,\ell y)\ell^r{}_k = C_{pqr}(x,\ell y)f^p{}_i f^q{}_k \ell^r{}_j. \tag{3.9}$$

Using (2.3), (3.9) is rewritten as

$$\begin{aligned} &C_{ijk}(x,\ell y) - C_{ijr}(x,\ell y)m^r{}_k \\ &= C_{pqj}(x,\ell y)f^p{}_i f^q{}_k - C_{pqr}(x,\ell y)f^p{}_i f^q{}_k m^r{}_j. \end{aligned} \tag{3.10}$$

Transvecting (3.10) with $f^i{}_t f^j{}_s \ell^k{}_h$ and using (2.3), we get

$$C_{ijk}(x,\ell y)f^i{}_t f^j{}_s \ell^k{}_h = -C_{pqj}(x,\ell y)\ell^p{}_t f^q{}_h f^j{}_s,$$

that is, $C_{ijk}(x,\ell y)f^i{}_t f^j{}_s \ell^k{}_h = 0$ by virtue of (3.8). Hence, from (3.6) we have $C_{ijr}(x,\ell y)\ell^r{}_k = \dot{\partial}_k g_{ij}(x,\ell y) = 0$, that is $g_{ij}(x,\ell y)$ is a Riemannian metric.

Thus we have

THEOREM 3.1. *If an $(f,L)$-manifold satisfies (3.5), then $g_{ij}(x,\ell y)$ is a Riemannian metric.*

## 4. An $(f,L)$-manifold with vanishing $h$-covariant derivative of $f$-structure

In an $(f,L)$-manifold, let $F\Gamma = (\overset{*}{\Gamma}{}^i{}_{jk}, G^i{}_j, C^i{}_{jk}), B\Gamma = (G^i{}_{jk}, G^i{}_j, 0)$ be the Cartan Finsler connection and the Berwald connection respectively, and let $\overset{*}{\nabla}_k, \overset{G}{\nabla}_k$ be the $h$-covariant derivatives with respect to $F\Gamma$ and $B\Gamma$ respectively, where $G^i{}_{jk} = \dot{\partial}_k G^i{}_j$.

We shall consider an $(f,L)$-manifold analogous with a Kaehlerian Finsler manifold. Let us assume that an $(f,L)$-manifold admits vanishing $h$-covariant derivative of the $f$-structure with respect to $F\Gamma$. Then we have

$$\overset{*}{\nabla}_k f^i{}_j = \partial_k f^i{}_j + \overset{*}{\Gamma}{}^i{}_{mk} f^m{}_j - f^i{}_m \overset{*}{\Gamma}{}^m{}_{jk} = 0.$$

From the relation $\overset{*}{\Gamma}{}^i{}_{km}y^m = G^i{}_k$, we get

$$y^m \partial_m f^i{}_j + G^i{}_m f^m{}_j - f^i{}_m G^m{}_j = 0. \tag{4.1}$$

Differentiating (4.1) with respect to $y^k$, we have

$$\overset{G}{\nabla}_k f^i{}_j = \partial_k f^i{}_j + G^i{}_{mk} f^m{}_j - f^i{}_m G^m{}_{jk} = 0. \tag{4.2}$$

If $\overset{*}{\nabla}_k f^i{}_j = 0$, we have from (2.2) and (4.2) $\overset{G}{\nabla}_k m^i{}_j = 0$, that is,

$$\partial_k m^i{}_j = -G^i{}_{rk} m^r{}_j + G^r{}_{jk} m^i{}_r.$$

Therefore, we have

$$\begin{aligned} &\partial_t m^h{}_s - \partial_s m^h{}_t \\ &= -G^h{}_{rt} m^r{}_s + G^r{}_{st} m^h{}_r + G^h{}_{rs} m^r{}_t - G^r{}_{ts} m^h{}_r = 0. \end{aligned} \tag{4.3}$$

From (2.4) and (4.3), the integrability condition of the distribution $\mathfrak{L}$ is satisfied.

Thus we have

THEOREM 4.1. *For an $(f, L)$-manifold, if the h-covariant derivative of f-structure with respect to $F\Gamma$ vanishes, then the distribution $\mathfrak{L}$ is integrable.*

Next, the Nijenhuis tensor constructed from the $f$-structure is given

$$N^h{}_{ij} = f^t{}_i \partial_t f^h{}_j - f^t{}_j \partial_t f^h{}_i - (\partial_i f^t{}_j - \partial_j f^t{}_i) f^h{}_t. \tag{4.4}$$

Substituting (4.2) in (4.4), we have

(4.5)

$$\begin{aligned} N^h{}_{ij} =& f^t{}_i(-G^h{}_{mt} f^m{}_j + f^h{}_m G^m{}_{jt}) \\ &- f^t{}_j(-G^h{}_{mt} f^m{}_i + f^h{}_m G^m{}_{it}) \\ &- (-G^t{}_{mi} f^m{}_j + f^t{}_m G^m{}_{ji} + G^t{}_{mi} f^m{}_i - f^t{}_m G^m{}_{ij}) f^h{}_t = 0. \end{aligned}$$

Thus, from (2.5), (4.5) and Theorem 4.1 we have

THEOREM 4.2. *For an $(f, L)$-manifold, if the $h$-covariant derivative of the $f$-structure vanishes, then the $f$-structure is partially integrable.*

Next, let $H_h{}^i{}_{jk}$ be the $h$-curvature tensor of $B\Gamma$. That is

$$H_h{}^i{}_{jk} = \delta_k G^i{}_{hj} + \delta_j G^i{}_{hk} + G^i{}_{rk} G^r{}_{hj} - G^i{}_{rj} G^r{}_{hk},$$

where $\delta_k = \partial_k - G^r{}_k \dot{\partial}_r$.

Applying the Ricci identity for $\overset{G}{\nabla}_k$ to $f^i{}_j$ [5], we have

$$H_r{}^h{}_{jk} f^r{}_i - f^h{}_r H_i{}^r{}_{jk} = 0. \tag{4.6}$$

Let $M^n$ be a constant curvature space, that is,

$$H_i{}^h{}_{jk} = K\{g_{ij}(x,y)\delta^h_k - g_{ik}(x,y)\delta^h_j\}. \tag{4.7}$$

Substituting (4.7) in (4.6), we have
(4.8)

$$K\{g_{rj}(x,y)\delta^h_k - g_{rk}(x,y)\delta^h_j\} f^r{}_i = f^h{}_r K\{g_{ij}(x,y)\delta^r_k - g_{ik}(x,y)\delta^r_j\}.$$

Let us assume that

$$g_{ir}(x,y) f^r{}_j + g_{jr}(x,y) f^r{}_j = 0. \tag{4.9}$$

If $f^i{}_j$ is an almost complex structure, then the condition (4.9) means that the Finsler metric becomes to the Riemannian metric [1], [3]. Here, since $f^i{}_j$ is an $f$-structure, we can see easily that the condition (4.9) reduces to $C_{ijk}(x.y) f^j{}_k = 0$, that is, the contraction of the derivative of Finsler metric $g_{jk}(x,y)$ with respect to $y^i$ and $f$-structure $f^j{}_h$ is a function of position alone.

Now, we suppose $K \neq 0$, then we have

$$g_{rj}(x,y) f^r{}_i \delta^h{}_k - g_{rk}(x,y) f^r{}_i \delta^h{}_j = f^h{}_k g_{ij}(x,y) - f^h{}_j g_{ik}(x.y). \tag{4.10}$$

Contracting (4.10) with respect to $h$ and $j$ and using the second equation of (2.4), we find

$$(1-n) g_{rk}(x,y) f^r{}_i = -g_{rk}(x,y) f^r{}_i,$$

from which $g_{rk}(x,y) f^r{}_i = 0$ for $n > 2$ by using (4.9). Therefore we have $f^i{}_j = 0$. That is a contraction. Consequently we obtain $K = 0$.

Thus we have

THEOREM 4.3. *Let $M^n(n > 2)$ be an $(f, L)$-manifold with constant curvature. If the $h$-covariant derivative of the $f$-structure with respect to the Cartan connection vanishes and satisfies* (4.9), *then the $h$-curvature tensor of the Berwald connection vanishes.*

REMARK. If the rank of $f$ is $n$ $(n > 2)$, then $f$ is an almost complex structure and the metric is a Riemannian one. Hence the $h$-curvature tensor $H_i{}^h{}_{jk}$ coincide with the Riemannian-Christoffel's curvature tensor. Thus Theorem 4.3 reduces to the well-known Bochner's theorem: *If a Kaehlerian manifold is of constant curvature, then it is of zero curvature.*

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