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HARDY-LITTLEWOOD MAXIMAL FUNCTIONS IN ORLICZ SPACES

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Abstract. Let $Mf(x)$ be the Hardy-Littlewood maximal function on $\mathbb{R}^n$. Let Φ and Ψ be functions satisfying $\Phi(t) = \int_0^t a(s)ds$ and $\Psi(t) = \int_0^t b(s)ds$, where $a(s)$ and $b(s)$ are positive continuous such that $\int_0^\infty \frac{a(s)}{s}ds = \infty$ and $b(s)$ is quasi-increasing. We show that if there exists a constant c_1 so that $\int_0^s \frac{a(t)}{t}dt \le c_1 b(c_1 s)$ for all $s \ge 0$, then there exists a constant c_1 such that

$$\int_{\mathbb{R}^n} \Phi(Mf(x))dx \le c_2 \int_{\mathbb{R}^n} \Psi(c_2|f(x)|)dx \tag{0.1}$$

for all $f \in L^1(\mathbb{R}^n)$. Conversely, if there exists a constant c_2 satisfying the condition (0.1), then there exists a constant c_1 so that $\int_\delta^s \frac{a(t)}{t}dt \le c_1 b(c_1 s)$ for all $\delta > 0$ and $s \ge \delta$.

1. Introduction

The Hardy-Littlewood maximal function $Mf(x)$ on $\mathbb{R}^n$ is defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy \tag{1.1}$$

where the supremum is taken over all open cubes $Q \subset \mathbb{R}^n$ with $x \in Q$.

The purpose of this paper is to give a necessary and sufficient conditions for Mf in terms of Orlicz space L^Φ. In [2], this problem is studied when f is given in unit circle.

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DEFINITION 1.1. Let $\Psi(t)$ be a nondecreasing continuous function such that $\lim_{t\to\infty}\Psi(t)=\infty$. Put

$$L^{\Psi}=\left\{f:\int_0^{\infty}\Psi(\epsilon|f(x)|)dx<\infty \text{ for some } \epsilon>0\right\}$$

Then the space L^{Ψ} is called an Orlicz space ([3] and [5]).

DEFINITION 1.2. Let $a(s)$ and $b(s)$ be positive continuous function defined on $[0,\infty)$ satisfying the following properties:

(i) $\int_0^{\infty}\frac{a(s)}{s}ds=\infty$.

(ii) $b(s)$ is quasi-increasing, that is, if there exists a constant c_o so that

$$b(s_1)\le c_o b(c_o s_2) \tag{1.2}$$

for all $0\le s_1\le s_2$. Define

(iii) $\Phi(t)=\int_0^t a(s)ds$ and $\Psi(t)=\int_0^t b(s)ds$ for $t>0$.

2. Main theorems

LEMMA 2.1. *If Ψ satisfies* (iii), *then $L^{\Psi}\subset L^1(\mathbb{R}^n)$.*

Proof. Since $b(s)$ is quasi-increasing, the following inequalities

$$\Psi(t)\ge\int_{t/2}^{t}b(s)ds\ge\frac{1}{c_o}\int_{t/2}^{t}b\left(\frac{t}{2c_o}\right)ds=\frac{t}{2c_o}b\left(\frac{t}{2c_o}\right)$$

implies $L^{\Psi}\subset L^1(\mathbb{R}^n)$. □

THEOREM 2.1. *Let $a(s)$, $b(s)$, $\Phi(t)$, and $\Psi(t)$ be functions satisfying the above properties* (i)-(iii). *If there exists a constant c_1 so that*

$$\int_0^s\frac{a(t)}{t}dt\le c_1 b(c_1 s) \tag{2.1}$$

for all $s\ge 0$, then there exists a constant c_2 such that

$$\int_{\mathbb{R}^n}\Phi(Mf(x))dx\le c_2\int_{\mathbb{R}^n}\Psi(c_2|f(x)|)dx \tag{2.2}$$

for all $f \in L^1(\mathbb{R}^n)$

Conversely, if there exists a constant c_1 *satisfying the condition* (2.2), *then there exists a constant* c_1 *so that for any* $\delta > 0$

$$\int_\delta^s \frac{a(t)}{t} dt \leq c_1 b(c_1 s) \tag{2.3}$$

for all $s \geq \delta$.

Proof. To prove (2.2), observe that

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(Mf(x))dx &= \int_0^\infty |\{\Phi(Mf(x)) > \lambda\}| d\lambda \\ &= \int_0^\infty |\{Mf(x) > \Phi^{-1}(\lambda)\}| d\lambda \\ &= \int_0^\infty |\{Mf(x) > t\}| a(t) dt. \end{aligned} \tag{2.4}$$

Since the maximal function M is simultaneously of weak type (1,1) and of type (∞, ∞) it follows that the well known result (page 92, Torchinsky [4]) that there exist constants c_3, c_4 such that

$$|\{Mf(x)) > t\}| \leq \frac{c_3}{t} \int_{t/c_4}^\infty |\{|f| > s\}| ds$$

for all $t > 0$. Hence it follows from Tonelli's theorem that

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(Mf(x))dx &= \int_0^\infty |\{Mf(x) > t\}| \Phi'(t) dt \\ &= \int_0^\infty |\{Mf(x) > t\}| a(t) dt \\ &= c_3 \int_0^\infty \frac{a(t)}{t} \left(\int_{t/c_4}^\infty |\{|f| > s\}| ds \right) dt \\ &= c_3 \int_0^\infty |\{|f| > s\} \left(\int_0^{c_4 s} \frac{a(t)}{t} dt \right) ds \\ &\leq c_1 c_3 \int_0^\infty |\{|f| > s\}| b(c_1 c_4 s) ds \end{aligned} \tag{2.5}$$

$$= \frac{c_3}{c_4} \int_0^\infty \left| \left\{ |f| > \frac{t}{c_1 c_4} \right\} \right| b(t) dt$$
$$= \frac{c_3}{c_4} \int_0^\infty |\{c_1 c_4 |f| > t\}| b(t) dt$$
$$= \frac{c_3}{c_4} \int_{\mathbb{R}^n} \Psi(c_1 c_4 |f(x)|) dx,$$

which proves (2.2).

Conversely, suppose that (2.2). If (2.3) does not hold, then there exist a sequence $\{s_n\}$ and $\delta > 0$ such that $s_k \geq 0$ for $k \geq 1$ and

$$\int_\delta^{s_k} \frac{a(t)}{t} > 2^k b(k 2^k s_k) \tag{2.6}$$

for all $k \geq 1$. Choose a collection of disjoint open cubes $\{Q_k\}$ so that

$$|Q_k| = \frac{1}{2^k \Psi(2^k s_k)} \tag{2.7}$$

and

$$\sum_{k=1}^\infty |Q_k| < \infty. \tag{2.8}$$

Put

$$f(x) = \frac{\epsilon_o}{c_2} \sum_{k=1}^\infty 2^k s_k \chi_{Q_k}, \tag{2.9}$$

where χ_{Q_k} is the characteristic function of Q_k and ϵ_o will be chosen in a moment. Then by (2.6) and (2.7) we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Psi(c_2 |f(x)|) dx &= \sum_{k=1}^\infty \int_{Q_k} \Psi(c_2 |f(x)|) dx \\ &= \sum \Psi(\epsilon_o 2^k s_k) |Q_k| \\ &\leq \sum_{k=1}^\infty \Psi(2^k s_k) \frac{1}{2^k \Psi(2^k s_k)} \\ &= \sum_{k=1}^\infty 2^{-k} < \infty. \end{aligned} \tag{2.10}$$

Since $L^\Psi \subset L^1(\mathbb{R}^n)$ by lemma 2.1, it follows that $f \in L^1(\mathbb{R}^n)$ and so $0 < ||f||_{L^1(\mathbb{R}^n)} < \infty$. Hence choose ϵ_o so that $||f||_{L^1(\mathbb{R}^n)} = 1$.

Now we will show that

$$\int_{\mathbb{R}^n} \Phi(Mf(x))dx = \infty. \tag{2.11}$$

But this leads to a contradiction, which will finish the proof. To show (2.11), put $g = \delta f$, where δ is given in (2.6). There exists a constant c so that

$$|\{Mg > \lambda\}| \geq \frac{c}{\lambda} \int_{|g|>\lambda} |g(x)|dx \tag{2.12}$$

for all $\lambda > ||g||_{L^1(\mathbb{R}^n)} = \delta$. (For this inequality, see Torchinsky [4], p. 93.) Hence by (2.11) and (2.12) we have

(2.13)

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi(\delta M(f(x))dx &= \int_{\mathbb{R}^n} \Phi(M(g(x))dx \\
&= \int_0^\infty |\{Mg > \lambda\}|\Phi'(\lambda)d\lambda \\
&\geq c\int_0^\infty \left(\int_{\{|g|>\lambda\}} |g(x)|dx\right) \frac{a(\lambda)}{\lambda} d\lambda \\
&= c\int_0^\infty \left(\int_{\mathbb{R}^n} |g(x)|\chi_{\{|g|>\lambda\}}(x)dx\right) \frac{a(\lambda)}{\lambda} d\lambda \\
&= \int_{\mathbb{R}^n} |g(x)| \left(\int_{||g||_{L^1(\mathbb{R}^n)}}^{|g(x)|} \frac{a(\lambda)}{\lambda} d\lambda\right) dx \\
&= \int_{\mathbb{R}^n} |g(x)| \left(\int_\delta^{|g(x)|} \frac{a(\lambda)}{\lambda} d\lambda\right) dx.
\end{aligned}$$

If $x \in I_k$, then $g(x) = \frac{\delta\epsilon_o}{c_2} 2^k s_k$. Thus from (2.6) it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi(M(g(x))dx &\geq c_5 \sum_{k=1}^\infty \int_{Q_k} |g(x)| \left(\int_\delta^{|g(x)|} \frac{a(\lambda)}{\lambda} d\lambda\right) dx \\
&= \frac{c_5 \epsilon\epsilon_o}{c_2} \sum_{k=1}^\infty 2^k s_k \left(\int_\delta^{\frac{\delta\epsilon_o 2^k s_k}{c_2}} \frac{a(\lambda)}{\lambda} d\lambda\right) |Q_k|
\end{aligned} \tag{2.14}$$

$$
\begin{aligned}
&\geq \frac{c_5\delta\epsilon_o}{c_2}\sum_{k=1}^{\infty} 2^k s_k \left(\int_\delta^{s_k} \frac{a(\lambda)}{\lambda} d\lambda\right) \frac{1}{2^k\Psi(2^k s_k)} \\
(2.14) \qquad &\geq \frac{c_5\delta\epsilon_o}{c_2}\sum_{k=1}^{\infty} 2^{2k} s_k b(k2^k s_k) \frac{1}{2^k\Psi(2^k s_k)}.
\end{aligned}
$$

Since $b(s)$ is quasi-increasing, we have

$$
\begin{aligned}
\Psi(2^k s_k) &= \int_0^{2^k s_k} b(s)ds \\
(2.15) \qquad &\leq \int_0^{2^k s_k} c_o b(c_o 2^k s_k) ds \\
&= c_o 2^k s_k b(c_o 2^k s_k)
\end{aligned}
$$

and so from (2.15) we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \Phi(\delta M(f(x)))dx &\geq \frac{c_5\delta\epsilon_o}{c_2}\sum_{k=1}^{\infty} \frac{b(k2^k s_k)}{b(c_o 2^k s_k)} \\
&\to \infty
\end{aligned}
$$

as $k \to \infty$. Since Φ is increasing, $\int_{\mathbb{R}^n} \Phi(Mf(x))dx = \infty$. This is what we needed. $\square$

If we define
$$\int_0^\infty \frac{a(t)}{t} dt = \lim_{\delta \downarrow 0} \int_\delta^\infty \frac{a(t)}{t} dt$$
then the following holds:

COROLLARY 1. *Let $a(s)$, $b(s)$, $\Phi(t)$, and $\Psi(t)$ be functions satisfying the above properties* (i)-(iii) *of* Definition 1.2. *Then following statements* (a) *and* (b) *are equivalent:*

(a) *There exists a constant c_1 so that*

$$\int_0^s \frac{a(t)}{t} dt \leq c_1 b(c_1 s)$$

for all $s \geq 0$.

(b) *There exists a constant* c_2 *such that*

$$\int_{\mathbb{R}^n} \Phi(Mf(x))dx \leq c_2 \int_{\mathbb{R}^n} \Psi(c_2|f(x)|)dx$$

for all $f \in L^1(\mathbb{R}^n)$.

COROLLARY 2. *Let* $a(s)$ *and* $\Phi(t)$ *be functions satisfying* (i-iii) *of* Definition 1.2. *Then following statements* (a) *and* (b) *are equivalent:*

(a) *There exists a constant* c_1 *so that*

$$\int_0^s \frac{a(t)}{t} dt \leq c_1 a(c_1 s)$$

for all $s \geq 0$.

(b) *There exists a constant* c_2 *such that*

$$\int_{\mathbb{R}^n} \Phi(Mf(x))dx \leq c_2 \int_{\mathbb{R}^n} \Phi(c_2|f(x)|)dx$$

for all $f \in L^1(\mathbb{R}^n)$.

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