Bull. Korean Math. Soc. **36** (1999), No. 2, pp. 239–246

# SOME REMARKS ON DING'S VANISHING THEOREM OF $\delta$-INVARIANT AND MONOMIAL CONJECTURE

KISUK LEE

ABSTRACT. We extend the Ding's vanishing theorem of $\delta$-invariant slightly on a Cohen-Macaulay local ring using the concept of Golod pairs. We also investigate the relation between the vanishing of $\delta$-invariant and Hochster's Monomial Conjecture.

## Introduction

A Cohen-Macaulay approximation was defined by Auslander and Buchweitz for a Gorenstein local ring $R$ as follows: Let $M$ be an $R$-module. An exact sequence of $R$-modules $0 \to Y_M \xrightarrow{\varphi} X_M \xrightarrow{\phi} M \to 0$ is called a Cohen-Macaulay approximation of $M$ if $\mathrm{projdim}_R Y_M < \infty$ and $X_M$ is a maximal Cohen-Macaulay module, i.e., $\mathrm{depth}\, X_M = \dim R$. The Cohen-Macaulay approximation is said to be minimal if all endomorphisms $\alpha : X_M \to X_M$ with $\phi \circ \alpha = \phi$ are isomorphisms ($\phi$ is called right minimal). It is known in [1,2] that a minimal Cohen-Macaulay approximation always exits. In [1], Auslander introduced the numerical invariant $\delta(M)$ for any finite module $M$ of a Gorenstein local ring $R$: $\delta(M)$ is defined to be the maximal rank of free summands of $X_M$ in a minimal Cohen-Macaulay approximation of $M$. Since then, the studies of the Auslander $\delta$-invariant have been done in various ways. For instances, the vanishing of $\delta$-invariant of syzygy modules of $R/\mathfrak{m}^i (i \geq 1)$ has been investigated and conjectured by Yoshino [14], and $\delta$-invariant was generalized to an arbitrary local ring by Martsinkovsky via Vogel's construction of Tate (co) homology [12]. In particular, Ding [3,4] has studied the $\delta$-invariant of cyclic modules $R/\mathfrak{m}^i (i \geq 1)$ and defined a new invariant, say *index*, of $R$. In [5, or see Corollary 1.7], he gave a sufficient

Received March 5, 1998.
1991 Mathematics Subject Classification: 13H10, 13C14, 13D02.
Key words and phrases: Auslander $\delta$-invariant, Golod pair, Monomial Conjecture.

condition for the vanishing of all $\delta(\Omega_R^i(M))$, where $\Omega_R^i(M)$ is denoted by the $i$-th syzygy module of an $R$-module $M$. The main goal of this paper is to extend his result slightly on a Cohen-Macaulay local ring using the concept of Golod pairs, which was defined by Gokhale [6].

In section 1, we first recall the definition of Golod pairs and results about Golod pairs, and then show the main result of this paper. More precisely, let $M$ be a finite module of a Cohen-Macaulay local ring $A$ and $I$ an ideal of $A$. We show that if $IM = 0$ and $(M, I)$ is a Golod pair, then $\delta$-invariant of a certain quotient module of every syzygy of $M$ over $A/I$ vanishes.

Section 2 deals with a remark on the relation between $\delta$-invariant and Hochster's Monomial Conjecture. In [9], Hochster himself proved that Monomial Conjecture is equivalent to Direct Summand Conjecture, which is known to be a central part among many homological conjectures because it implies most of them [10]. The full Monomial Conjecture is still open even though the conjecture has been proved for several kinds of rings, for examples, rings containing a field, or of dimension less than 3 [9]. Unfortunately, we do not complete Monomial Conjecture in this paper. Instead, we set a question about the vanishing of $\delta$-invariant of a certain quotient ring, which implies Monomial Conjecture.

All rings in this paper will be assumed to be commutative Noetherian and all modules are unitary.

The author is grateful to J. Koh for pointing out the relation between $\delta$-invariant and Monomial Conjecture.

## 1. Vanishing theorem of $\delta$-invariant

Gokhale [6] has introduced a new construction of a projective resolution of a module over a quotient ring. More precisely, let $I$ be an ideal of a Noetherian local ring $A$ and $M$ a finite $A/I$-module. Then a free resolution $(\bar{U}_\bullet, \bar{h}_\bullet)$ of $M$ over $A/I$ has been constructed from resolutions $(Q_\bullet, e_\bullet)$ and $(P_\bullet, d_\bullet)$ of $M$ and $I$, respectively, over $A$, where $U_0 = Q_0$, $U_1 = Q_1$, $U_n = Q_n \amalg (\amalg_{i=0}^{n-2}(P_{n-i-2} \otimes_s U_i))$ and $\bar{U}_i = A/I \otimes_s U_i$.

In general, the resolution $\bar{U}_\bullet$ is not necessarily minimal. This leads to the definition of a Golod pair. We say that $(M, I)$ is a *Golod pair* over $A$ if the constructed $A/I$-resolution $\bar{U}_\bullet$ of $M$ is minimal. In the case $M = A/\mathfrak{m}$, this definition of Golod pair is the same as the previously known concepts (e.g., Golod ideals, Golod homomorphisms, etc.). Using mapping cones, Kwong described a different way of building the Gokhale's resolution, which gives a better understanding of the maps in the resolution. He extended many results in Golod ideals to the case of Golod pairs in [11].

For a finite module $M$ on a Gorenstein local ring $R$, $\delta(M)$ can be also characterized without using a Cohen-Macaulay approximation as follows: $\delta(M)$ is the smallest integer $n$ such that there is an epimorphism $X \coprod A^n \to M$ with a maximal Cohen-Macaulay module $X$ with no free summand [8, Lemma 1.1]. This fact leads to the extension of $\delta$-invariant to the modules on Cohen-Macaulay local rings. Let $M$ be a finite module of a Cohen-Macaulay local ring $A$. $\delta(M)$ is defined to be 0 if $M$ is a homomorphic image of a maximal Cohen-Macaulay module without free summand. Otherwise, $\delta(M) \neq 0$ [3].

Using a concept of Golod pair, we extend slightly the Ding's vanishing theorem [5] of the Auslander $\delta$-invariant on a certain Cohen-Macaulay local ring.

First, we state some facts, which are used later.

PROPOSITION 1.1. [1] *Let $R$ be a Gorenstein local ring and $N$, $M$ finite $A$-modules. Then*
(a) *If* $\mathrm{projdim}_R M < \infty$, *then* $\delta(M) > 0$.
(b) *If $N \to M \to 0$ is an epimorphism, then $\delta(N) \geq \delta(M)$.*

THEOREM 1.2. [SHAMASH] [6] *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $M$ be a finite $A$-module. If $x \in \mathfrak{m}\mathrm{Ann}(M)$, where $x$ is a non zero-divisor of $A$, then $(M, xA)$ is a Golod pair.*

THEOREM 1.3. ([6], Theorem 2.5) *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Suppose $(M, I)$ is a Golod pair and $(\bar{U}_\bullet, \bar{h}_\bullet)$ is a minimal $A/I$-projective resolution of $M$. Then for $n \geq 0$, $(\Omega^{n+1}_{A/I}(M), I)$ is also a Golod pair, where $\Omega^{n+1}_{A/I}(M)$ is the $(n+1)$-th $A/I$-syzygy of $M$, i.e., the image of $\bar{h}_{n+1}$.*

Throughout this paper $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring and $I$ is an ideal of $A$. Let $M$ be a finite $A$-module such that $IM = 0$. Thus $M$ is a finite $R := A/I$-module. We denote by $\Omega^i_R(M)$ (respectively, $\Omega^i_A(M)$)

the $i$-th syzygy of $M$ in the minimal resolution of $M$ over $R$ (respectively, over $A$), and $\Omega_R^0(M) = M$ . By $\mu(N)$ we mean the minimal number of generators of a finite module $N$ as an $A$-module.

LEMMA 1.4. *If $(M, I)$ is a Golod pair, then* $\mu(\Omega_R^1(M)) = \mu(\Omega_A^1(M))$.

*Proof.* Let $\cdots \to A^{n_1} \xrightarrow{d_1} A^{n_0} \xrightarrow{d_0} M \to 0$ be a minimal resolution of $M$ over $A$. Then since $(M, I)$ is a Golod pair, by Proposition 2.3 in [6] we have

$$IA^{n_0} \subseteq \mathfrak{m} \cdot \operatorname{Im} d_1.$$

Now consider the diagram,

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & & IA^{n_0} & & \Omega_A^1(M) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & A^{n_0} & \xrightarrow{Id} & A^{n_0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_R^1(M) & \to & A^{n_0}/IA^{n_0} & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then since $IA^{n_0} \subseteq \operatorname{Ker} d_0 = \operatorname{Im} d_1 = \Omega_A^1(M)$, by Snake Lemma we have

$$0 \to IA^{n_0} \hookrightarrow \Omega_A^1(M) \to \Omega_R^1(M) \to 0.$$

Thus we have

$$IA^{n_0} \subseteq \mathfrak{m}\Omega_A^1(M) \quad \text{if and only if} \quad \mu(\Omega_A^1(M)) = \mu(\Omega_R^1(M)).$$

Since $IA^{n_0} \subseteq \mathfrak{m} \cdot \operatorname{Im} d_1 = \mathfrak{m}\Omega_A^1(M)$, we have $\mu(\Omega_A^1(M)) = \mu(\Omega_R^1(M))$. □

Lemma 1.5 below is shown in the proof of Theorem 2.1 in [5], but we state the properties in the form that is convenient for us to quote. We also include a proof for reader's convenience.

Let $0 \to \Omega_A^1(M) \to A^{n_0} \to M \to 0$ be a projective cover of $M$ over $A$. We have the following exact sequence by tensoring with $R = A/I$:

$$0 \to \operatorname{Tor}_1^A(M, R) \to \Omega_A^1(M) \otimes R \to \Omega_R^1(M) \to 0.$$

Now we consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \Omega_R^2(M) & \to & R^{m_1} & \to & \Omega_R^1(M) & \to 0 \\ & \downarrow\epsilon & & \downarrow & & \| & \\ 0 \to & \operatorname{Tor}_1^A(M,R) & \to & \overline{\Omega_A^1(M)} & \to & \Omega_R^1(M) & \to 0, \end{array}$$

where $\cdots \to R^{m_1} \to R^{m_0} \to M \to 0$ is a minimal resolution of $M$ over $R$, and $\overline{\Omega_A^1(M)} := \Omega_A^1(M) \otimes R$.

LEMMA 1.5. *In the situation just described above, the following conditions are equivalent:*
(1) $\epsilon : \Omega_R^2(M) \longrightarrow \operatorname{Tor}_1^A(M,R)$ *is an epimorphism;*
(2) $\mu(\Omega_R^1(M)) = \mu(\Omega_A^1(M))$.

*Proof.* We note that $\epsilon$ is an epimorphism if and only if $\phi : R^{m_1} \to \overline{\Omega_A^1(M)}$ is an epimorphism by Five Lemma. Suppose $\phi$ is surjective. Then $m^1 \geq \mu(\Omega_A^1(M)) \geq \mu(\Omega_R^1(M)) = m^1$, and so the equality holds. Conversely, if $\mu(\Omega_A^1(M)) = \mu(\Omega_R^1(M))$, then $\phi \otimes R/\mathfrak{m}_R$ is an isomorphism, and so $\phi$ is surjective. □

Now we are ready to prove our main theorem.

THEOREM 1.6. *We assume that $R = A/I$ is Cohen-Macaulay. If $(M, I)$ is a Golod pair, then for any $i \geq 0$ we have*

$$\delta(\Omega_R^i(M)/J\Omega_R^i(M)) = 0,$$

*where $J = I_1(\varphi)$, i.e., an ideal generated by $1 \times 1$-minors of $\varphi$ which is the presenting matrix of $I$ over $A$,* $A^{n_1} \xrightarrow{\varphi} A^{n_0} \to I \to 0$.

*Proof.* We know $\Omega_R^2(M) \longrightarrow \operatorname{Tor}_1^A(M,R)$ is an epimorphism by Lemma 1.4, and Lemma 1.5. Since

$\operatorname{Tor}_1^A(M,R) = H_1(M^{n_1} \xrightarrow{\varphi} M^{n_0} \xrightarrow{0} M) = M^{n_0}/\operatorname{Im}\varphi$ and $\operatorname{Im}\varphi \subseteq JM^{n_0}$,

we have an epimorphism,

$$\Phi : \Omega_R^2(M) \to \operatorname{Tor}_1^A(M,R) \to M^{n_0}/JM^{n_0} \to M/JM.$$

By Theorem 2.2, we know that $(\Omega_R^2(M), I)$ is also a Golod pair.
Thus by replacing $M$ by $\Omega_R^2(M)$, we have an epimorphism
$\Omega_R^4(M) = \Omega_R^2(\Omega_R^2(M)) \to \Omega_R^2(M)/J\Omega_R^2(M)$. Since $J\Omega_R^2(M) \subseteq \ker\Phi$, we

have an epimorphism $\Omega_R^2(M)/J\Omega_R^2(M) \to M/JM$. By continuing this process, there exists an epimorphism, for $j \geq 0$,

$$\Omega_R^{2j+2}(M) \to \Omega_R^{2j}(M)/J\Omega_R^{2j}(M) \to \cdots \to \Omega_R^2(M)/J\Omega_R^2(M) \to M/JM.$$

Since $R = A/IA$ is a Cohen-Macaulay ring, $\Omega_R^i(M)$ is a maximal Cohen-Macaulay module with no free summand for $i > \dim R$. Hence we have for any $i \geq 0$

$$\delta(\Omega_R^{2i}(M)/J\Omega_R^{2i}(M)) = 0.$$

Starting with $\Omega_R^1(M)$ instead of $M$, the case of odd $i$ follows. □

COROLLARY 1.7. ([5], Theorem 2.1) *Let $(A, \mathfrak{m}_A)$ be a Gorenstein local ring and $M$ a finite $A$-module. If $x \in \mathfrak{m}_A \mathrm{Ann}(M)$ is a non zero-divisor, then $\delta(\Omega_R^i(M)) = 0$ for all $i \geq 0$, where $R = A/xA$.*

*Proof.* By Shamash Theorem (Theorem 1.2), we know that $(M, xA)$ is a Golod pair. Since $0 \xrightarrow{\varphi} A \to xA \to 0$ and $J = I_1(\varphi) = 0$, by the above theorem, we have the conclusion. □

## 2. Relation between $\delta$-invariant and monomial conjecture

Let us first briefly describe Monomial Conjecture. For more background on this conjecture, [9] and [10] are recommended.

**Monomial Conjecture (MC)** : Let $A$ be a $d$-dimensional Noetherian local ring with a system of parameters $x_1, \cdots, x_d$. Then there exists no equation

$$x_1^t \cdots x_d^t = a_1 x_1^{t+1} + \cdots + a_d x_d^{t+1} \quad \text{with all } a_i \in A \text{ and all integer } t\ .$$

Thus to prove **(MC)**, it suffices to show that

$$x_1^t \cdots x_d^t : A/(x_1, \cdots, x_d) \to A/(x_1^{t+1}, \cdots, x_d^{t+1})$$

is not a zero-map for any integer $t$. The direct limit of this system is known to be a $d$-th local cohomology $H_{\mathfrak{m}}^d(A)$, which is not zero by local duality [7]. Finally, it is enough to show that $\mu_A^{\mathrm{x}} : A/(x_1, \cdots, x_d) \to H_{\mathfrak{m}}^d(A)$ is not a zero-map.

In [13], for a Gorenstein local ring $R$ and an ideal $I$ of $R$ consisting of zero-divisors, it was shown that

$$\mu_A^{\mathbf{x}} \neq 0 \text{ if and only if } \mathrm{Ann}_R I \not\subseteq (y_1, \cdots, y_d),$$

where $y_1, \cdots, y_d$ is a system of parameters of $R$ and $x_1, \cdots, x_d$ is a corresponding system of parameters of $A = R/I$. Since we may assume that $A$ is complete, using Cohen Structure Theorem we may conclude that showing the necessary condition in the above for any complete intersection ring $R$ implies **MC**. Hence by Proposition 1.1, we know that $\delta(R/\mathrm{Ann}_R I) = 0$ implies **MC**. Since $I$ consists of zero-divisors, there exists a minimal prime $P$ of $R$ such that $I \subseteq P$. Since $\mathrm{Ann}_R P \subseteq \mathrm{Ann}_R I$, by Proposition 1.1 $\delta(R/\mathrm{Ann}_R P) = 0$ implies $\delta(R/\mathrm{Ann}_R I) = 0$. Thus we may raise the following question:

QUESTION 2.1. Let $R$ be a complete intersection ring and $P$ a minimal prime of $R$. Then under what conditions does $\delta(R/\mathrm{Ann}_R P) = 0$ hold?

REMARK 2.2. If $R$ is a 0-,or 1-dimensional complete intersection ring, then for any minimal prime $P$ of $R$, $\delta(R/\mathrm{Ann}_R P) = 0$ since $R/\mathrm{Ann}_R P$ is a maximal Cohen-Macaulay ring without free summand. Indeed, dimension 0 case is clear. Suppose that $\dim R = 1$. Let $P = (x_1, \cdots, x_t)$ and $\mathrm{Ann}_R P = (y_1, \cdots, y_s)$. Consider the following exact sequence:

$$\cdots \longrightarrow R^s \xrightarrow{(y_1,\cdots,y_s)^T} R \xrightarrow{(x_1,\cdots,x_t)} R^t \longrightarrow N \to 0,$$

where $(y_1, \cdots, y_s)^T$ is a transpose map of $(y_1, \cdots, y_s)$ and $N$ is a cokernel of a map $(x_1, \cdots, x_t)$. Then $R/\mathrm{Ann}_R P$ is the first syzygy of $N$, and so $R/\mathrm{Ann}_R P$ is a maximal Cohen-Macaulay module with no free summand(trivial) since $\dim R = 1$. In general if $\operatorname{depth} N \geq \dim R - 1$, then $R/\mathrm{Ann}_R P$ is a maximal Cohen-Macaulay module with no free summand.

## References


[1] M. Auslander, *Minimal Cohen-Macaulay approximations,* preprint.

[2] M. Auslander and R. O. Buchweitz, *The homological theory of maximal Cohen-Macaulay approximation,* Mem. Soc. Math. France **38** (1989), 5-37.

[3] S. Ding, *A note on the index of Cohen-Macaulay local rings,* Comm. Alg. **21** (1993), 53-71.

[4] ———, *Auslander's δ-invariants of Gorenstein local rings,* Proc. Amer. Math. Soc. **122** (1994), 649-656.

[5] ———, *Cohen-Macaulay approximation and multiplicity,* J. Alg. **153** (1992), 271-288.

[6] D. Gokhale, *Resolutions mod I and Golod pairs,* Comm. Alg. **22** (1994), 989-1030.

[7] A. Grothendieck, *Local Cohomology,* Lect. Notes. in Math. 41, Springer-Verlag, New York, 1967.

[8] J. Herzog, *On the index of a homogeneous Gorenstein ring,* Contem. Math. **159** (1994), 95-102.

[9] M. Hochster, *Contracted ideals from integral extensions of regular rings,* Nagoya Math. J. **51** (1973), 25-43.

[10] ———, *Topics in the homological theory of modules over commutative rings,* CBMS Lect. Notes 24, Amer. Math. Soc., Providence, 1975.

[11] C. Kwong, *Mapping cones construction and Golod Pairs,* Ph. D. Thesis, Indiana University, 1997.

[12] Martsinkovsky, *A remarkable property of the (co)syzygy modules of the residue field of a nonregular local ring,* J. Pure Appl. Alg. **110** (1996), 9-13.

[13] J. R. Strooker and J. Stückrad, *Monomial conjecture and complete intersections,* Manuscr. Math **79** (1993), 153-159.

[14] Y. Yoshino, *On the higher delta invariants of a Gorenstein local ring,* Proc. Amer. Math. Soc. **124** (1996), 2641-2647.

DEPARTMENT OF MATHEMATICS, SOOKMYUNG WOMEN'S UNIVERSITY, SEOUL 140-742, KOREA
*E-mail*: kilee@sookmyung.ac.kr