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ON NON-PROPER PSEUDO-EINSTEIN RULED REAL HYPERSURFACES IN COMPLEX SPACE FORMS

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ABSTRACT. In the paper [12] we have introduced the new kind of *pseudo-Einstein* ruled real hypersurfaces in complex space forms $M_n(c)$, $c\neq 0$, which are foliated by *pseudo-Einstein* leaves. The purpose of this paper is to give a geometric condition for non-proper *pseudo-Einstein* ruled real hypersurfaces to be *totally geodesic* in the sense of Kimura [8] for $c > 0$ and Ahn, Lee and the present author [1] for $c < 0$.

1. Introduction

A complex $n(\geq 2)$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according as $c > 0$, $c = 0$ or $c < 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g).

There exist many studies about real hypersurfaces of $M_n(c)$. One of the first research is the classification of homogeneous real hypersurfaces in a complex projective space $P_n(\mathbb{C})$ by Takagi [14], who showed that these hypersurfaces of $P_n(\mathbb{C})$ could be divided into six types which are said to be of type A_1, A_2, B, C, D, and E, and in Cecil-Ryan [4] and Kimura [7] proved that they are realized as the tubes of constant

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radius over Kaehlerian submanifolds if the structure vector field ξ is principal. Also Berndt [2,3] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_n(\mathbb{C})$ are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal. Nowadays in $H_n(\mathbb{C})$ they are said to be of type A_0, A_1, A_2, and B.

When the structure vector field ξ is not principal, Kimura [8] and Ahn, Lee and the present author [1] have constructed an example of ruled real hypersurfaces foliated by totally geodesic leaves, which are integrable submanifolds of the distribution T_0 defined by the subspace $T_0(x) = \{X{\in}T_xM | X{\perp}\xi, x{\in}M\}$, along the direction of ξ and *Einstein* complex hypersurfaces in $P_n(\mathbb{C})$ and $H_n(\mathbb{C})$ respectively. The expression of the Weingarten map is given by

$$A\xi = \alpha\xi + \beta U, \ AU = \beta\xi \text{ and } AX = 0,$$

where we have defined a unit vector U orthogonal to ξ in such a way that $\beta U = A\xi - \alpha\xi$ and β denotes the length of a vector field $A\xi - \alpha\xi$ and $\beta(x){\neq}0$ for any point x in M, and for any X in the distribution T_0 and orthogonal to ξ. Recently, several characterizations of such kind of ruled real hypersurfaces have been studied by the papers ([1], [3], [8], [9] and [13]). Moreover, among them there are so many ruled real hypersurfaces, which are foliated in *parallel* by the leaves of the distribution $T_0 = \{X{\in}T_xM | X{\perp}\xi\}$ along the integral curve of the structure vector ξ. Then in such a situation the vector field U defined in above is always *parallel* along the direction of ξ.

Now as a general extension of this fact we introduce a new kind of ruled real hypersurfaces in $M_n(c)$ foliated by pseudo-Einstein leaves, which are integrable submanifolds of the distribution T_0 defined by the subspace $\{X{\in}T_xM | X{\perp}\xi\}$, along the direction of ξ and *pseudo-Einstein* complex hypersurfaces in $M_n(c)$. Then such kind of ruled real hypersurfaces are said to be *pseudo-Einstein*, because its Ricci tensor of the integral submanifold $M(t)$ is given by

$$S^t = (\frac{n}{2}c - \mu)I + (\mu - \lambda)\{U{\otimes}U^* + \phi U{\otimes}(\phi U)^*\}.$$

Moreover, its expression of the Weingarten map is given by

$$AU = \beta\xi + \gamma U + \delta\phi U \text{ and } \quad A\phi U = \delta U - \gamma\phi U.$$

In Lemma 3.1 we know that the function λ in above is given by $\lambda = 2(\gamma^2 + \delta^2)$. When $\lambda = \mu$, ruled real hypersurfaces foliated by such kind of leaves are said to be *Einstein.* In particular, $\lambda = \mu = 0$, this kind of Einstein ruled real hypersurfaces are congruent to ruled real hypersurfaces in $M_n(c)$ foliated by totally geodesic Einstein leaves $M_{n-1}(c)$, which are said to be *totally geodesic* ruled real hypersurfaces in the sense of Kimura [8] for $c > 0$ and Ahn, Lee and the present author [1] for $c < 0$. In such a situation the function γ and δ both vanish identically.

When the function $\mu = 0$ and at least one of the functions γ and δ vanishes identically, this kind of *pseudo-Einstein* ruled ones are said to be *non-proper.* Of course, totally geodesic ruled ones in the sense of Kimura [8] and Ahn, Lee and the present author [1] are contained in the class of *non-proper* pseudo-Einstein ruled real hypersurfaces.

Then it naturally rises to the question that "Whether this kind of non-proper *pseudo-Einstein* ruled real hypersurfaces in $M_n(c)$ except totally geodesic ruled ones can be existed or not ?" Or otherwise, "What kind of geometric condition can be imposed for non-proper pseudo Einstein ruled ones to be congruent to one of geodesic ruled ones ?" From this point of view we answer this problem affirmatively and assert the following:

THEOREM. *Let M be a non-proper pseudo-Einstein ruled real hypersurface in $M_n(c)$, $c\neq 0$, $n\geq 2$. If the vector U is parallel along the direction of ξ, then M is locally congruent to one of ruled real hypersurfaces with each leaves totally geodesics and parallel along the direction of the structure vector field ξ.*

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2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let C be a unit normal vector field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the transformation of X and C under J can be represented as

$$JX = \phi X + \eta(X)C, \qquad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$,where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Accordingly, the set is so called an *almost contact metric structure.* Furthermore the covariant derivative of the structure tensors are given by

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX, \tag{2.1}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M.

Since the ambient space is of constant holomorphic sectional curvature c, the equation of Gauss and Codazzi are respectively given as follows

$$\begin{aligned} &R(Y, Z)X \\ &= \frac{c}{4}\{g(Z, X)Y - g(Y, X)Z + g(\phi Z, X)\phi Y - g(\phi Y, X)\phi Z \\ &\quad - 2g(\phi Y, Z)\phi X\} + g(AZ, X)AY - g(AY, X)AZ, \end{aligned} \tag{2.2}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X. Now let us suppose that the structure vector ξ is a principal vector with principal curvature α, that is, $A\xi = \alpha\xi$. Then, differentiating this, we have

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX, \tag{2.4}$$

where we have used (2.1). Then it follows

$$g((\nabla_X A)Y, \xi) = (X\alpha)\eta(Y) + \alpha g(Y, \phi AX) - g(Y, A\phi AX) \tag{2.5}$$

for any tangent vector fields X and Y on M. By the equation of Codazzi (2.3), we have

$$2A\phi AX - \frac{c}{2}\phi X = \alpha(\phi A + A\phi)X. \tag{2.6}$$

Now in order to get our results, we introduce a lemma, which was derived from the formulas (2.4), (2.5) and (2.6), in the paper [5] due to Ki and the present author as follows:

LEMMA 2.1. *Let M be a real hypersurface in a complex space form $M_n(c)$, $n \geq 2$. If it satisfies*

$$A\phi + \phi A = 0, \tag{2.7}$$

then we have $c = 0$.

3. Pseudo-Einstein ruled real hypersurface

This section is concerned with the necessary properties about *pseudo-Einstein ruled* real hypersurfaces. Before going to give the notion of pseudo-Einstein ruled ones, we recall a ruled real hypersurface M of $M_n(c)$, $c \neq 0$ which is defined in Kimura [7]. Let us denote by $\mathcal{D}$ a J-invariant integrable $(2n-2)$-dimensional distribution defined on $M_n(c)$ whose integral manifolds are holomorphic planes normal to the plane spanned by unit normals C and JC and let $\gamma : I \rightarrow M_n(c)$ be an integral curve for the vector $\xi = -JC$.

For any $t(\in I)$ let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x \in M^{(t)}_{n-1}(c) : t \in I\}$. Then the construction of M asserts that M is a real hypersurface of $M_n(c)$, which is called a ruled real hypersurface. This means that there exists a ruled real hypersurfaces of $M_n(c)$ with the given distribution $\mathcal{D}$. This kind of ruled real hypersurface is foliated by leaves, which are totally geodesic complex hypersurfaces $M^{(t)}_{n-1}(c)$. Then from its construction it can be easily seen that the expression of the Weingarten map is given by

$$A\xi = \alpha\xi + \beta U, \ AU = \beta\xi \text{ and } AX = 0,$$

where U is a unit vector orthogonal to ξ and α and β ($\beta \neq 0$) denote certain differentiable functions defined on M and for any X in $\mathcal{D}$ orthogonal to U. Moreover, it can be easily seen that the Ricci tensor S^t of the complex hypersurface $M(t)$ in $M_n(c)$ is proportional to its Riemannian metric such that $S^t = \frac{nc}{2}g$. That is, all of its leaves are Einstein complex hypersurfaces in $M_n(c)$. So such a ruled real hypersurface is naturally said to be *Einstein ruled.*

Now let us consider more generalized notion than the above ones. We want to consider a generalized ruled real hypersurface M, which is foliated by *pseudo-Einstein* leaves. Here, the meaning of *pseudo-Einstein* leaves are integrable submanifolds of the distribution $\mathcal{D}$ which are *pseudo-Einstein* complex hypersurfaces in $M_n(c)$. Then in this case, this kind of generalized ruled real hypersurface is said to be *pseudo-Einstein ruled* real hypersurfaces.

For the construction of this, let us also consider a regular curve $\gamma : I \to M_n(c)$. Then for any $t (\in I)$ let $\Gamma^{(t)}_{n-1}$ be a *pseudo-Einstein* complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x \in \Gamma^{(t)}_{n-1} : t \in I\}$. Then this construction gives us many *pseudo-Einstein* ruled real hypersurface.

Now, let us consider two shape operators A_C and A_ξ of any integral submanifold $M(t) = \Gamma^{(t)}_{n-1}$ of the distribution $\mathcal{D} = \{X \in T_xM | X \perp \xi\}$ in $M_n(c)$ in the direction of C and ξ. For any unit vector field V along $\mathcal{D}$, let V^* be the corresponding 1-form defined by $V^*(V) = g(V, V) = 1$.

If they satisfy

$$A_\xi^2 + A_C^2 = \mu I + \lambda(V \otimes V^* + \phi V \otimes (\phi V)^*)$$

for a certain vector field V, where λ and μ are smooth function on M, then the real hypersurface M with the given distribution $\mathcal{D}$ of $M_n(c)$ is said to be *pseudo-Einstein ruled.* In particular, if $\lambda = \mu$, then it is said to be *Einstein ruled* and if $\lambda = \mu = 0$, then it is said to be *totally geodesic* and *Einstein ruled*, and is the ruled real hypersurface as discussed in above. Accordingly, we say that the real hypersurface M is *pseudo-Einstein ruled, Einstein ruled* or *totally geodesic ruled*, then it is easily seen that any integral submanifold of $\mathcal{D}$, which is a submanifold of real codimension 2 in $M_n(c)$, is *pseudo-Einstein, Einstein* or *totally geodesic*, respectively.

Since $T_0(= \mathcal{D})$ is integrable, we know that

$$g((A\phi + \phi A)X, Y) = 0 \tag{3.1}$$

for any vector fields X and Y in T_0 (See Kimura [8], Kimura and Maeda [9]).

Now we are going to give a Ricci tensor of the integral submanifold $M(t)$ of the distribution $\mathcal{D}$, which is a *pseudo-Einstein* submanifold of real codimension 2 in $M_n(c)$. Since $M(t)$ is a submanifold of codimension 2, ξ and C are orthonormal vector fields on its leaf in $M_n(c)$. So by the equation of Gauss, we have

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + g(AX, Y)C \\ &= \nabla_X^t Y + g(A_\xi X, Y)\xi + g(A_C X, Y)C,\end{aligned}$$

where $\bar{\nabla}$ and ∇^t are the covariant derivatives in the ambient space $M_n(c)$ and in the submanifold $M(t)$, respectively and moreover A_C and A_ξ are the shape operators in the direction of C and ξ, respectively. Then we have

$$g(\bar{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = g(A_\xi X, Y),$$

for any $X, Y \in T_0$, from which it implies that

$$A_\xi X = -\phi AX, \ X \in T_0. \tag{3.2}$$

On the other hand, by the equation of Gauss, we have

$$g(AX, Y) = g(A_C X, Y), \ X, Y \in T_0$$

and therefore

$$A_C X = AX - \beta g(X, U)\xi, \ X \in T_0. \tag{3.3}$$

By (3.1) we have a formula

$$A\phi X = -\phi AX - \beta g(X, \phi U)\xi, \ X \in T_0. \tag{3.4}$$

From this it can be easily seen that the traces of these two shape operators A_ξ and A_C are both equal to zero. Now by using (2.2), the curvature tensor of the integral submanifold $M(t)$ is given by

$$\begin{aligned} &g(R^t(X,Y)Z,W) \\ &= \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\phi Y,Z)g(\phi X,W) \\ &\quad - g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W)\} \\ &\quad + g(A_\xi Y,Z)g(A_\xi X,W) + g(A_C Y,Z)g(A_C X,W) \\ &\quad - g(A_\xi X,Z)g(A_\xi Y,W) - g(A_C X,Z)g(A_C Y,W) \end{aligned}$$

for any vector fields X, Y, Z and W in $\mathcal{D}$. Since the traces of the above two shape operators A_ξ and A_C are both equal to zero, its Ricci tensor S^t of $M(t)$ in $M_n(c)$ is given by

$$\begin{aligned} g(S^t Y, Z) &= \sum_{i=1}^{2n-2} g(R^t(e_i, Y)Z, e_i) \\ &= \frac{n}{2}cg(Y,Z) - g((A_\xi^2 + A_C^2)Y, Z) \end{aligned} \tag{3.5}$$

for any Y, Z in $\mathcal{D}$. Then the tensor $A_\xi^2 + A_C^2$ of the pseudo-Einstein submanifold $M(t)$ can be constructed in such a way that

$$(3.6) \qquad \begin{cases} (A_\xi^2 + A_C^2)U = & \lambda U, \\ (A_\xi^2 + A_C^2)\phi U = & \lambda\phi U, \\ (A_\xi^2 + A_C^2)X = & \mu X, \quad X\in\mathcal{D}\perp U, \phi U, \end{cases}$$

where λ and μ are smooth functions on $M(t)$. So its Ricci tensor S^t of $M(t)$ is given by

$$S^t = \left(\frac{n}{2}c - \mu\right) I + (\mu - \lambda)\{U\otimes U^* + \phi U\otimes(\phi U)^*\}.$$

Then from the formula (3.5) it follows

LEMMA 3.1. (See [12]) *Let M be a pseudo-Einstein and not Einstein ruled real hypersurfaces in $M_n(c)$, $c\neq 0$, $n\geq 3$. Then we have*

$$(3.7) \qquad \begin{cases} AU = & \beta\xi + \gamma U + \delta\phi U, \\ A\phi U = & \delta U - \gamma\phi U, \quad \lambda = 2(\gamma^2 + \delta^2) \end{cases}$$

In particular, if it is totally geodesic, we have $\gamma = \delta = 0$.

Proof. Naturally let us put

$$(3.8) \qquad \begin{cases} A\xi = & \alpha\xi + \beta U, \\ AU = & \beta\xi + \gamma U + \delta\phi U + \epsilon X, \\ A\phi U = & -\gamma\phi U + \delta U - \epsilon\phi X, \end{cases}$$

for some vector field X orthogonal to ξ, U and ϕU where in the third equation we have used the condition (3.1), because the distribution $\mathcal{D}$ is integrable. Since M is supposed to be proper *pseudo-Einstein*, we may put $\lambda\neq\mu$. In order to prove $\epsilon = 0$, firstly let us prove the following

$$(3.9) \qquad A^2U = (\alpha + \gamma)\beta\xi + \left(\beta^2 + \frac{\lambda}{2}\right) U.$$

Indeed, (3.2), (3.3) and the first formula of (3.6) imply

$$\begin{aligned}\lambda U &= -A_\xi \phi AU + A_C(AU - \beta\xi)\\ &= \phi A\phi AU + A(AU - \beta\xi) - \beta g(AU - \beta\xi, U)\xi\\ &= 2\{A^2U - \beta A\xi - \beta g(AU, U)\xi\},\end{aligned}$$

where in the third equality we have used the condition (3.4). Secondly, we calculate the following

$$A^2\phi U = \beta\delta\xi + \frac{\lambda}{2}\phi U. \tag{3.10}$$

In fact, (3.2), (3.3) and the second formula of (3.6) give

$$\begin{aligned}\lambda\phi U =&(A_\xi^2 + A_C^2)\phi U\\ =&\phi A^2U + A^2\phi U - \beta^2\phi U - \beta g(A\phi U, U)\xi.\end{aligned}$$

So by (3.8) we get the above (3.10). Finally we give the following for any X orthogonal to ξ, U and ϕU.

$$A^2X = \beta\epsilon\xi + \frac{\mu}{2}X, \tag{3.11}$$

because the third formula of (3.6) and the condition (3.1) imply that

$$\begin{aligned}\mu X =& - A_\xi\phi AX + A_C\{AX - \beta g(X, U)\xi\}\\ =&2(A^2X - \beta g(AX, U)\xi).\end{aligned}$$

Now let us apply the shape operator A to the second formula of (3.8) and use also (3.8) and (3.9). Then

$$\begin{aligned}\epsilon AX &= \left(\frac{\lambda}{2} - \gamma^2 - \delta^2\right)U - \gamma\epsilon X + \delta\epsilon\phi X\\ &= \epsilon^2U - \gamma\epsilon X + \delta\epsilon\phi X,\end{aligned}$$

where we have used

$$\|A\phi U\|^2 = \gamma^2 + \delta^2 + \epsilon^2 = \frac{\lambda}{2}, \tag{3.12}$$

which can be obtained from (3.8) and (3.10). So let us assume $\epsilon \neq 0$, then $AX = \epsilon U - \gamma X + \delta\phi X$. This implies

$$\begin{aligned} A^2X =& \epsilon AU - \gamma AX + \delta A\phi X \\ =& (\beta\xi + \gamma U + \delta\phi U + \epsilon X) - \gamma(\epsilon U - \gamma X + \delta\phi X) \\ & - (\epsilon\phi U - \gamma\phi X - \delta X) \\ =& \epsilon\beta\xi + (\epsilon^2 + \gamma^2 + \delta^2)X. \end{aligned}$$

From this together with (3.11) it follows

$$\mu = 2(\gamma^2 + \delta^2 + \epsilon^2).$$

Then by (3.12) we have $\lambda = \mu$, which makes a contradiction. So we should have $\epsilon = 0$. It completes the proof of Lemma 3.1. □

REMARK 3.1. When both the functions λ and μ in (3.6) vanish identically, (3.10) and (3.11) imply respectively

$$A\phi U = 0 \text{ and } AX = 0$$

for any X orthogonal to ξ, U and ϕU. Then from this together with (3.12) it follows the function $\lambda = 0$, that is $\gamma = \delta = 0$. Then naturally, M is congruent to totally geodesic ruled real hypersurfaces in $M_n(c)$, $c \neq 0$.

REMARK 3.2. When the function μ in (3.6) vanishes identically, (3.11) gives $\|AX\| = 0$. This implies $\epsilon = 0$. So it naturally satisfies

$$\begin{cases} AU &= \beta\xi + \gamma U + \delta\phi U, \\ A\phi U &= \delta U - \gamma\phi U, \\ AX &= 0, \quad X \perp \xi, U, \phi U. \end{cases}$$

When $\mu=0$ and at least one of the function γ and δ vanishes identically, M is said to be non-proper *pseudo-Einstein* ruled real hypersurfaces. So for convenience sake let us say the function δ vanishes identically. Then by Remark 3.2 we can put

$$(3.13)\qquad \begin{cases} A\xi &= \alpha\xi+\beta U,\\ AU &= \beta\xi+\gamma U,\\ A\phi U &= -\gamma\phi U,\\ AX &= 0 \end{cases}$$

for any $X\in T_1$, where T_1 denotes a distribution defined by a subspace $T_1(x)=\{u\in T_0(x):g(u,U(x))=g(u,\phi U(x))=0\}$.

Next the covariant derivative $(\nabla_X A)Y$ with respect to X and Y in T_0 is explicitly expressed. The equation (2.3) of Codazzi gives us

$$(\nabla_X A)\xi-(\nabla_\xi A)X=-\frac{c}{4}\phi X.$$

By the direct calculation of the left hand side of the above relation and using the second equation of (3.2) we get

$$(3.14)\qquad \begin{aligned} &d\alpha(X)\xi+\alpha\phi AX+d\beta(X)U+\beta\nabla_X U-A\phi AX\\ &\quad-\nabla_\xi(AX)+A\nabla_\xi X+\frac{c}{4}\phi X=0,\ X\in T_0. \end{aligned}$$

Now from (3.1), (3.2) and the above equation we can derive the following

LEMMA 3.2. *Let M be a non-proper pseudo-Einstein ruled real hypersurfaces in $M_n(c)$, $c\neq 0$, $n\geq 2$. Then it follows*

$$(3.15)\qquad \beta\nabla_X U=\begin{cases} \{\beta^2-\gamma^2-\alpha\gamma-\frac{c}{4}+2g(\nabla_\xi U,\phi U)\gamma\}\phi U, & X=U\\ -(\xi\gamma)\phi U-\gamma\phi\nabla_\xi U, & X=\phi U\\ -\frac{c}{4}\phi X-g(X,\phi\nabla_\xi U)\gamma\phi U, & X\in T_1, \end{cases}$$

and

$$(3.16)\qquad d\beta(X)=\begin{cases} 0, & X=U\\ \gamma^2-\alpha\gamma+\beta^2+\frac{c}{4}-2\gamma g(\nabla_\xi\phi U,U), & X=\phi U\\ -\gamma g(\nabla_\xi X,U), & X\in T_1. \end{cases}$$

Proof. Putting $X = U$ in (3.14) and taking an inner product with U imply

$$d\beta(U) = 0.$$

Moreover, by taking an orthogonal part from ξ and U, we know that

$$\beta\nabla_U U + \alpha\phi AU - A\phi AU + \frac{c}{4}\phi U - \beta^2\phi U + A\nabla_\xi U - \gamma\nabla_\xi U = 0.$$

From this, together with (3.13), it follows that

$$\beta\nabla_U U = \left\{\beta^2 - \gamma^2 - \alpha\gamma - \frac{c}{4} + 2g(\nabla_\xi U, \phi U)\gamma\right\}\phi U.$$

So we have the first formula of (3.15). Also putting $X = \phi U$ in (3.14) and using (3.13), we have

$$\begin{aligned} d\alpha(\phi U)\xi + \alpha\gamma U + d\beta(\phi U)U + \beta\nabla_{\phi U}U - A\phi A\phi U \\ + \nabla_\xi(\gamma\phi U) + A\nabla_\xi\phi U = -\frac{c}{4}\phi^2 U. \end{aligned} \tag{3.17}$$

On the other hand, we can put

$$\nabla_\xi U = g(\nabla_\xi U, \phi U)\phi U + g(\nabla_\xi U, Z)Z$$

for some unit Z in $T_1 = [\xi, U, \phi U]^\perp$ which is the orthogonal complement of the distribution $[\xi, U, \phi U]$ spanned by ξ,U and ϕU. So by (3.13), it follows that

$$A\phi\nabla_\xi U = -g(\nabla_\xi U, \phi U)AU.$$

Then the last two terms of the left side of (3.17) becomes

$$\begin{aligned} \nabla_\xi(\gamma\phi U) + A\nabla_\xi(\phi U) = d\gamma(\xi)\phi U + \gamma\{-\beta\xi + \phi\nabla_\xi U\} \\ - g(\nabla_\xi U, \phi U)AU - \beta\{\alpha\xi + \beta U\}. \end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.17) and comparing an orthogonal part from ξ and U imply

$$\beta\nabla_{\phi U}U = -d\gamma(\xi)\phi U - \gamma\phi\nabla_\xi U.$$

So, this gives the second formula of (3.15). Moreover, if we take an inner product (3.17) with U, we also get the second formula of (3.16).

On the other hand, (3.13) and (3.14) imply

$$d\alpha(X)\xi + d\beta(X)U + \beta\nabla_X U + A\nabla_\xi X + \frac{c}{4}\phi X = 0 \tag{3.19}$$

for any $X \in T_1$. From this, if we take an inner product with U, it follows

$$d\beta(X) = -\gamma g(\nabla_\xi X, U),$$

which gives the third formula of (3.16).

Now for any $X \in T_1$ we can express the vector $\nabla_\xi X$ in such a way that

$$\nabla_\xi X = g(\nabla_\xi X, U)U + g(\nabla_\xi X, \phi U)\phi U + g(\nabla_\xi X, Z)Z$$

for some unit Z in T_1. Then by applying the shape operator A to this formula and substituting this into (3.19), we have

$$\begin{aligned} &d\alpha(X)\xi + d\beta(X)U + \beta\nabla_X U + g(\nabla_\xi X, U)AU \\ &\quad + g(X, \phi\nabla_\xi U)\gamma\phi U + \frac{c}{4}\phi X = 0. \end{aligned}$$

From this, if we compare the part orthogonal to ξ and U, we have

$$\beta\nabla_X U = -\frac{c}{4}\phi X - g(X, \phi\nabla_\xi U)\gamma\phi U, \quad X \in T_1.$$

Accordingly, we have completed the proof of Lemma 3.2. □

Now let us calculate

$$\begin{aligned} &g((\nabla_X A)Y, \xi) \\ &= g((\nabla_X A)\xi, Y) = g(\nabla_X(A\xi) - A\nabla_X\xi, Y) \\ &= g((X\alpha)\xi + \alpha\nabla_X\xi + (X\beta)U + \beta\nabla_X U - A\phi AX, Y) \\ &= \alpha g(\phi AX, Y) + d\beta(X)g(Y, U) + \beta g(\nabla_X U, Y) - g(A\phi AX, Y). \end{aligned}$$

Now any $X \in T_0$ can be written in such a way that

$$X = g(X,U)U + g(X,\phi U)\phi U + g(X,Z)Z \tag{3.20}$$

for some unit $Z \in T_1$. Then for any X in T_1 this expression and Lemma 3.2 imply the followings:

$$\begin{aligned}\beta\nabla_X U =& \beta g(X,U)\nabla_U U + \beta g(X,\phi U)\nabla_{\phi U}U + g(X,Z)\beta\nabla_Z U \\ =& g(X,U)\{\beta^2 - \gamma^2 - \alpha\gamma - \frac{c}{4} + 2\gamma g(\nabla_\xi U, \phi U)\}\phi U \\ &+ g(X,\phi U)\{-d\gamma(\xi)\phi U - \gamma\phi\nabla_\xi U\} \\ &+ g(X,Z)\{-\frac{c}{4}\phi Z - \gamma g(Z,\phi\nabla_\xi U)\phi U\},\end{aligned}$$

and

$$\begin{aligned}d\beta(X) =& g(X,\phi U)\{\gamma^2 - \alpha\gamma + \beta^2 + \frac{c}{4} - 2\gamma g(\nabla_\xi \phi U, U)\} \\ &- \gamma g(X,Z)g(\nabla_\xi Z, U).\end{aligned}$$

So it follows that

(3.21)

$$\begin{aligned}& g((\nabla_X A)Y,\xi) \\ =& d\beta(X)g(U,Y) + \beta g(\nabla_X U, Y) + \alpha g(\phi AX, Y) \\ &- g(A\phi AX, Y) \\ =& \big[g(X,U)\{\beta^2 - \gamma^2 - \alpha\gamma - \frac{c}{4} + 2\gamma g(\nabla_\xi U, \phi U)\} \\ &- g(X,\phi U)d\gamma(\xi) - \gamma g(X,Z)g(Z,\phi\nabla_\xi U)\big]g(\phi U, Y) \\ &+ \{-\gamma g(X,\phi U)g(\phi\nabla_\xi U, Y) - \frac{c}{4}g(X,Z)g(\phi Z, Y)\} \\ &+ \{\gamma^2 - \alpha\gamma + \beta^2 + \frac{c}{4} - 2\gamma g(\nabla_\xi\phi U, U)\}g(X,\phi U)g(U,Y) \\ &- \gamma g(X,Z)g(\nabla_\xi Z, U)g(U,Y) + \alpha g(\phi AX, Y) - g(A\phi AX, Y).\end{aligned}$$

On the other hand, the expression (3.20) and (3.13) implies

$$\phi AX = \gamma g(X,U)\phi U + \gamma g(X,\phi U)U,$$

and

$$\begin{aligned} A\phi AX =&\gamma g(X,U)A\phi U + \gamma g(X,\phi U)AU \\ =& -\gamma^2 g(X,U)\phi U + \gamma g(X,\phi U)(\beta\xi + \gamma U) \\ =& -\gamma^2 g(X,U)\phi U + \gamma\beta g(X,\phi U)\xi + \gamma^2 g(X,\phi U)U. \end{aligned}$$

Substituting these formulas into (3.21) and taking account of the equation of Codazzi (2.3), we have for any $X, Y \in T_0$ and $Z \in T_1$

$$\begin{aligned} &g((\nabla_\xi A)X, Y) \\ &= g((\nabla_X A)Y, \xi) \\ &= \beta^2\{g(X,U)g(\phi U, Y) + g(X,\phi U)g(U,Y)\} \\ &\quad + 2\gamma g(\nabla_\xi U, \phi U)\{g(X,U)g(\phi U,Y) + g(X,\phi U)g(U,Y)\} \\ &\quad - g(X,\phi U)g(Y,\phi U)d\gamma(\xi) - \gamma g(X,Z)g(Z,\phi\nabla_\xi U)g(\phi U, Y) \\ &\quad - \gamma g(X,\phi U)g(\phi\nabla_\xi U, Y) - \gamma g(X,Z)g(\nabla_\xi Z, U)g(U,Y), \end{aligned}$$

where we have used the fact that

$$\begin{aligned} &-\frac{c}{4}\{g(X,U)g(\phi U,Y) - g(X,\phi U)g(U,Y) + g(X,Z)g(\phi Z, Y)\} \\ &= -\frac{c}{4}g(\phi X, Y) \end{aligned}$$

because of (3.20). From this we can assert

$$(\nabla_\xi A)U = \{\beta^2 + 2\gamma g(\nabla_\xi U, \phi U)\}\phi U + \theta(U)\xi,$$

and

$$\begin{aligned} (\nabla_\xi A)\phi U =&\beta^2 U + 2\gamma g(\nabla_\xi U, \phi U)U \\ &- d\gamma(\xi)\phi U - \gamma\phi\nabla_\xi U + \theta(\phi U)\xi. \end{aligned}$$

Now summing up the formulas in above, non-proper pseudo Einstein ruled real hypersurfaces in $M_n(c)$ satisfy the followings:

(3.22)

$$\begin{cases} (\nabla_\xi A)U & \equiv \lambda\phi U \pmod{\xi}, \\ (\nabla_\xi A)\phi U & \equiv \lambda U - d\gamma(\xi)\phi U - \gamma\phi\nabla_\xi U \pmod{\xi}, \\ (\nabla_\xi A)X & \equiv -\gamma g(X,\phi\nabla_\xi U)\phi U + \gamma g(X,\nabla_\xi U)U \pmod{\xi}. \end{cases}$$

When we assume that the vector field U is parallel along the direction ξ, then $\nabla_\xi U = 0$ and $\gamma = g(AU, U)$ is constant along the direction ξ. Then (3.22) together with this assumption imply that a non-proper pseudo Einstein ruled real hypersurface satisfies

$$(3.23) \qquad (\nabla_\xi A)X \equiv f\phi AX,$$

where $f\gamma = \beta^2$.

4. Proof of the Theorem

In this section we want to prove the main Theorem. It will be turned out that *non-proper pseudo Einstein* ruled real hypersurfaces in complex space form $M_n(c), c\neq 0$ are only totally geodesic ruled real hypersurfaces in $M_n(c)$ foliated in such a way that its structure vector U is parallel along the direction of ξ. Namely it will be congruent to one of ruled real hypersurfaces in th sense of Kimura [8] for $c > 0$ and Ahn, Lee and Suh [1] for $c < 0$.

Let M be a real hypersurface in complex space forms $M_n(c)$, $c\neq 0$. Now let us denote by T_0 be a distribution defined by the subspace

$$T_0(x) = \{X \in T_x M | g(X, \xi_x) = 0\}$$

in the tangent space $T_x M$ at any point x in M, which is called a *holomorphic distribution.*

On the other hand, we have seen in section 3 that *non-proper pseudo Einstein* ruled real hypersurfaces in $M_n(c)$ with the vector field U is parallel along the direction of ξ satisfies

$$(4.1) \qquad (\nabla_\xi A)X \equiv f\phi AX \pmod{\xi}$$

for any vector field X in T_0 and a smooth function f without zero points. Moreover, we have known that its structure vector ξ is not principal.

First of all, we assert the following

LEMMA 4.1. *Let M be a real hypersurface satisfying*

$$(\nabla_\xi A)X \equiv f\phi AX \quad (mod\ \xi), \tag{4.1}$$

for any vector field X in T_0 and a smooth function f without zero points, then the distribution T_0 is integrable.

Proof. By the assumption (4.1) and

$$g((\nabla_\xi A)X, Y) = g((\nabla_\xi A)Y, X)$$

it turns out to be

$$fg((A\phi + \phi A)X, Y) = 0$$

for any vector fields X and Y in T_0. Since the function f has no zero points,

$$g((A\phi + \phi A)X, Y) = 0 \tag{4.2}$$

for any vector fields X and Y in T_0. It completes the proof. □

Let M be a real hypersurface in $M_n(c)$, $c\neq 0$, $n\geq 2$ satisfying (4.1). Then the distribution T_0 is integrable by Lemma 4.1.

Now we can put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in the holomorphic distribution T_0, and α and β are smooth functions on M. So we may consider that the function β does not vanish identically on M. Let M_0 be the non-empty open subset of M consisting of points x at which $\beta(x)\neq 0$. Moreover, the set M_0 is a dense subset of M.

In fact, we suppose that the interior of the subset $M - M_0$ is not empty. On the interior we see $\beta = 0$. Namely, ξ is a principal vector with principal curvature α. Then by (4.2) we have

$$A\phi + \phi A = 0.$$

Since Lemma 2.1 is a local property, it implies $c = 0$ on the interior and hence on the whole M, a contradiction. So the interior of the subspace $M - M_0$ is empty, namely M_0 is dense in M.

By (2.3) and (4.1) there is a 1-form θ such that

$$(\nabla_Y A)\xi = f\phi AY - \frac{1}{4}c\phi Y + \theta(Y)\xi$$

for any vector field Y in T_0.

Differentiating $A\xi = \alpha\xi + \beta U$ covariantly with respect to any vector field X in T_0, we have

$$\begin{aligned}\beta\nabla_X U =& A\phi AX + (f-\alpha)\phi AX - \frac{1}{4}c\phi X \\ & - d\alpha(X)\xi - d\beta(X)U + \theta(X)\xi,\end{aligned} \tag{4.3}$$

where we have used (4.1) and (4.2). By (4.2) the above equation can be reformed as

$$\begin{aligned}\beta\nabla_X U =& -\phi A^2 X + (f-\alpha)\phi AX - \frac{1}{4}c\phi X \\ & + \{-d\alpha(X) - \beta g(AX, \phi U) + \theta(X)\}\xi - d\beta(X)U.\end{aligned}$$

From this, if we take an inner product of the above equation with ξ, we get

$$-d\alpha(X) - \beta g(AX, \phi U) + \theta(X) = \beta g(AX, \phi U).$$

Thus it follows

$$\begin{aligned}\beta\nabla_X U =& -\phi A^2 X + (f-\alpha)\phi AX - \frac{1}{4}c\phi X \\ & + \beta g(AX, \phi U)\xi - d\beta(X)U\end{aligned} \tag{4.4}$$

for any vector field X in T_0.

On the open subset M_0 we can put $AU = \beta\xi + \gamma U + \delta V$, where ξ, U and V are orthonormal vector fields, and γ and δ are smooth functions on M_0. Now let us denote by $L(V, W, \cdots, X, Y)$ a subspace in the tangent subspace T_xM spanned by any vectors $V, W, \cdots, X, Y$ at any point x. Then by Lemma 4.1 we also have the following.

LEMMA 4.2. *Let M be a real hypersurface in $M_n(c)$, $n\geq 3$. If it satisfies*

$$(4.1) \qquad (\nabla_\xi A)X \equiv f\phi AX \pmod{\xi}$$

for any vector field X in T_0 and a smooth function f without zero points, then $L(\xi, A\xi)$ is not A-invariant.

Proof. By Lemma 2.1 and the above remark we know that on the subset M_0 the vector $A\xi$ can be expressed as $A\xi = \alpha\xi + \beta U$, where β is a smooth function defined on M and U is a unit vector orthogonal to ξ.

Now let us suppose that the linear subspace $L(\xi, A\xi)$ in the tangent space of M is A- *invariant.* Then the vector AU can be written in such a way that $AU = \beta\xi + \gamma U$. From this, together with the integrability of the distribution T_0 in Lemma 4.1, we have

$$(4.5) \qquad A\phi U = -\gamma\phi U,$$

because $(A\phi + \phi A)U = 0$. Differentiating $A\xi = \alpha\xi + \beta U$ along X in T_0, by the assumption (4.1) we also have the formula (4.3). Then taking an inner product (4.3) with ϕU and using (4.5) imply

$$(4.6) \qquad \beta g(\nabla_X U, \phi U) = (f - \alpha - \gamma)g(AX, U) - \frac{1}{4}cg(X, U).$$

Taking an inner product of (4.4) with ϕU, we obtain

$$\beta g(\nabla_X U, \phi U) = -g(A^2X, U) + (f - \alpha)g(AX, U) - \frac{1}{4}cg(X, U)$$

for any vector field X in T_0. From (4.5) and the above equation it follows that

$$g(A^2X, U) = \gamma g(AX, U).$$

So it implies

$$\beta g(AX, \xi) = 0$$

for any vector field X in T_0, a contradiction. Thus we have the conclusion. □

Now we are going to prove our main Theorem in introduction. Let M be a non-proper pseudo-Einstein ruled real hypersurfaces in $M_n(c)$. Then it satisfies

$$(*)\quad \begin{cases} (\nabla_\xi A)U & \equiv \lambda\phi U \pmod{\xi} \\ (\nabla_\xi A)\phi U & \equiv \lambda U - dr(\xi)\phi U - \gamma\phi\nabla_\xi U \pmod{\xi} \\ (\nabla_\xi A)X & \equiv -g(X,\phi\nabla_\xi U)\gamma\phi U + \gamma g(X,\nabla_\xi U)U \pmod{\xi}. \end{cases}$$

When we assume that U is parallel along the direction of ξ, then $\nabla_\xi U = 0$ and the smooth function γ is constant along the direction of ξ.

Now let us suppose that the function γ has no zero points. Then by the assumption $\nabla_\xi U = 0$ and the formula (*), we know that non-proper pseudo-Einstein ruled real hypersurfaces in $M_n(c)$ satisfy (4.1) for a smooth function f which has no zero points in such a way that

$$(\nabla_\xi A)X \equiv f\phi AX \pmod{\xi},\ f = \frac{\beta^2}{\gamma}.$$

Then Lemma 4.2 implies that $L(\xi, A\xi)$ is not *A-invariant.* But in section 3 we know that $AU = \beta\xi + \gamma U$ for a non-proper pseudo Einstein ruled real hypersurfaces. This makes a contradiction. So we should have that the function γ has some zero points.

Now let us denote by M' a subset in M consisting of points at which γ has the value 0. That is, the set $M' = \{x \in M | \gamma(x) = 0\}$ should be non-empty. Now we suppose that $M - M' \neq \phi$. Then on $M - M'$ we know that the function γ has no zero points. Accordingly, by using the same arguments as in above, we can also makes a contradiction. So we should have $M - M' = \phi$. That is, the set M' is dense in M. Then by the continuity, the function γ vanishes identically on M. This means that M is totally geodesic pseudo-Einstein ruled real hypersurface in the sense of Kimura [8] for $c > 0$ and Ahn, Lee and the present author [1] for $c < 0$. Consequently, we complete the proof of our Theorem.

References

[1] S. S. Ahn, S. B. Lee and Y. J. Suh, *On ruled real hypersurfaces in a complex space form*, Tsukuba J. Math. **17** (1993), 311–322.

[2] J. Berndt, *Real hypersurfaces with constant principal curvatures in a complex hyperbolic space*, J. Reine Angew. Math. **395** (1989), 132–141.
[3] ———, *Geometry and Topology of submanifolds* II, (Avignon, 1988), 10-19, World Scientific, 1990.
[4] T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.
[5] U-H. Ki and Y. J. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama **32** (1990), 207–221.
[6] ———, *On a characterization of real hypersurfaces of type A in a complex space form*, Canadian Math. Bull. **37** (1994), 238–244.
[7] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans Amer. Math. Soc. **296** (1986), 137–149.
[8] ———, *Sectional curvatures of a holomorphic plane in $P_n(C)$*, Math. Ann. **276** (1987), 487–497.
[9] M. Kimura and S. Maeda, *On real hypersurfaces of a complex projective space*, Math. Z. **202** (1989), 299–311.
[10] Y. Maeda, *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan **28** (1976), 529–540.
[11] S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan **37** (1985), 515–535.
[12] Y. J. Suh, *On pseudo Einstein ruled real hypersurfaces in complex space forms*, to appear in Note di Matematica **19** (1999).
[13] ———, *Characterizations of real hypersurfaces in complex space forms in terms of Weingarten map*, Nihonkai Math. J., **6** (1995), 63–79.
[14] R. Takagi, *On homogeneous real hypersurfaces of a complex projective space*, Osaka J. Math. **10** (1973), 495–506.

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