Bull. Korean Math. Soc. **36** (1999), No. 2, pp. 351–355

A DOUBLY COMMUTING PAIR OF HYPONORMAL OPERATORS

Yong Tae Kim

Abstract. If (H_1, H_2) is a doubly commuting pair of hyponormal operators on a Hilbert spaces $\mathcal{H}$, then there exists a commuting pair (T_1, T_2) of contractions on $\mathcal{H}$ such that $H_i = H_i^* T_i$ for each $i = 1, 2$.

Throughout this note let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called a *contraction* if $||T|| \leq 1$ and is called *hyponormal* if $T^*T - TT^* \geq 0$. A pair (T_1, T_2) of operators on $\mathcal{H}$ is called a *doubly commuting pair* if $T_1T_2 = T_2T_1$ and $T_1T_2^* = T_2^*T_1$. In 1966, R. Douglas ([**2**]) showed that if $A, B \in \mathcal{L}(\mathcal{H})$ then the following are equivalent:

(i) $A = BC$ for some $C \in \mathcal{L}(\mathcal{H})$;
(ii) $||A^*x|| \leq k\,||B^*x||$ for some $k \geq 0$ and all $x \in \mathcal{H}$;
(iii) $\operatorname{ran} A \subseteq \operatorname{ran} B$.

As an interesting corollary of the above theorem, it can be shown (cf. [**1**]) that if H is a hyponormal operator on $\mathcal{H}$ then there exists a contraction $T \in \mathcal{L}(\mathcal{H})$ such that $H = H^*T$. If (H_1, H_2) is a commuting pair of hyponormal operators on $\mathcal{H}$ then there exist contractions $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ such that $H_i = H_i^*T_i$ for each $i = 1, 2$. But in this case, T_1 and T_2 may not commute. In this note we show that there exist commuting such contractions for a doubly commuting pair of hyponormal operators. The following is our main theorem.

Received June 27, 1998.
1991 Mathematics Subject Classification: Primary 47B20; Secondary 47A13.
Key words and phrases: hyponormal operators, doubly commuting, contractions.
This paper was supported by Faculty Research Fund, Sungkyunkwan University, 1997

THEOREM 1. *If (H_1, H_2) is a doubly commuting pair of hyponormal operators on a Hilbert space $\mathcal{H}$ then there exists a commuting pair (T_1, T_2) of contractions on $\mathcal{H}$ such that $H_i = H_i^* T_i$ for each $i = 1, 2$.*

Proof. Suppose that for $i = 1, 2$, H_i is a hyponormal operator on $\mathcal{H}$. Then for each $x_i \in \mathcal{H}$, there is a unique $y_i \in \mathrm{cl}\,(\mathrm{ran}\, H_i)$ with

$$H_i x_i = H_i^* y_i. \tag{1}$$

Write $T_i x_i = y_i$. Then T_i is a contraction on $\mathcal{H}$ satisfying (cf. [**2**])

$$H_i = H_i^* T_i \quad \text{and} \quad \mathrm{ran}\, T_i \subseteq \mathrm{cl}\,(\mathrm{ran}\, H_i) \qquad (i = 1, 2). \tag{2}$$

We will show that $T_1 T_2 = T_2 T_1$ if (H_1, H_2) is a doubly commuting pair. Suppose (H_1, H_2) is a doubly commuting pair. By (2) we have

$$\begin{aligned} H_1 H_2 &= H_1 H_2^* T_2 = H_2^* H_1 T_2 = H_2^* H_1^* T_1 T_2 = H_1^* H_2^* T_1 T_2 \\ H_2 H_1 &= H_2 H_1^* T_1 = H_1^* H_2 T_1 = H_1^* H_2^* T_2 T_1, \end{aligned} \tag{3}$$

which gives

$$H_1 (H_2^* T_1 T_2) = H_1^* (H_2^* T_2 T_1). \tag{4}$$

We now claim that

$$H_2^* T_1 T_2 (\mathcal{H}) \subseteq \mathrm{cl}\,(\mathrm{ran}\, H_1) \tag{5}$$

and

$$H_2^* T_2 T_1 (\mathcal{H}) = H_2 T_1 (\mathcal{H}) \subseteq \mathrm{cl}\,(\mathrm{ran}\, H_1). \tag{6}$$

Indeed, we have, for each $x \in \mathcal{H}$

$$\begin{aligned} H_2^* T_1 T_2 x &= H_2^* (\lim H_1 y_n) \qquad \text{with } \lim H_1 y_n = T_1 T_2 x \quad \text{(by (2))} \\ &= \lim H_2^* H_1 y_n \\ &= \lim H_1 H_2^* y_n \in \mathrm{cl}\,(\mathrm{ran}\, H_1) \end{aligned}$$

and

$$\begin{aligned} H_2^*T_2T_1x &= H_2T_1x \\ &= H_2\big(\lim H_1z_n\big) \qquad \text{with } \lim H_1z_n = T_1x \\ &= \lim H_2H_1z_n \\ &= \lim H_1H_2z_n \in \operatorname{cl}(\operatorname{ran} H_1), \end{aligned}$$

which gives (5) and (6). But since $\operatorname{cl}(\operatorname{ran} H_1) = (\ker H_1^*)^\perp$, it follows that

$$H_2^*T_1T_2 = H_2^*T_2T_1. \tag{7}$$

Next we will show that

$$T_2T_1(\mathcal{H}) \subseteq \operatorname{cl}(\operatorname{ran} H_2) \quad \text{and} \quad T_1T_2(\mathcal{H}) \subseteq \operatorname{cl}(\operatorname{ran} H_2). \tag{8}$$

The first part of (8) follows at once from (2). For the second part of (8) suppose that for each $x \in \mathcal{H}$,

$$y := T_1T_2x = T_1\big(\lim H_2x_n\big) \qquad \text{with } y \in \operatorname{cl}(\operatorname{ran} H_1),$$

where the second equality is guaranteed by (2). It now suffices to show that $y \in \operatorname{cl}(\operatorname{ran} H_2)$. Indeed we have

$$\begin{aligned} H_1^*y &= H_1\big(\lim H_2x_n\big) \qquad \text{(by (1))} \\ &= \lim H_1H_2x_n \\ &= \lim H_2H_1^*T_1x_n \\ &= H_1^*\big(\lim H_2T_1x_n\big). \end{aligned}$$

Since by (6), $\lim H_2T_1x_n \in \operatorname{cl}(\operatorname{ran} H_1)$, it follows that $y = \lim H_2T_1x_n \in \operatorname{cl}(\operatorname{ran} H_2)$, which gives the second part of (8). By (7) and (8), we can conclude that $T_1T_2 = T_2T_1$. □

REMARK 2. Even though H_1 and H_2 are commuting hyponormal operators, the product H_1H_2 need not be hyponormal: in fact, there exists a hyponormal operator whose square is not hyponormal (cf. [**3**, Solution 209]). However the equality (3) in the proof of Theorem 1 shows that if H_1 and H_2 are doubly commuting hyponormal operators then the product H_1H_2 is also hyponormal.

On the other hand Sz.-Nagy's theorem on the dilation theory is as follows (cf. [**4**]): *Every contraction $T \in \mathcal{L}(\mathcal{H})$ has a dilation to an isometry V on $\ell_2(\mathcal{H})$ given by*

$$V = \begin{pmatrix} T & 0 & 0 & 0 & 0 & \cdots \\ D_T & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix},$$

where $D_T = (I - T^*T)^{1/2}$. Ando's work, which applies to the case of two commuting contraction, is as follows (cf. [**4**]): *Every pair of commuting contractions T_1, T_2 has a dilation to a pair of commuting isometries V_1, V_2 on $\ell_2(\mathcal{H})$.* Thus we can have:

COROLLARY 3. *If (H_1, H_2) is a doubly commuting pair of hyponormal operators on a Hilbert space $\mathcal{H}$ then there exists a commuting pair (V_1, V_2) of isometries on $\ell_2(\mathcal{H})$ such that $H_i = H_i^* P_{\mathcal{H}} V_i|_{\mathcal{H}}$ $(i = 1, 2)$, where $P_{\mathcal{H}}$ denotes the orthogonal projection from $\ell_2(\mathcal{H})$ onto $\mathcal{H}$.*

Proof. This follows at once from Theorem 1 and the preceding remark. □

References

[1] J. B. Conway, *Subnormal Operators*, Pitman, Boston, 1981.

[2] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.

[3] P. R. Halmos, *A Hilbert Space Problem Book*, Springer, New York, 1982.

[4] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert spaces*, Acadèmiai Kiad ò-North Holland, Budapest, 1955.

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
E-mail: ytkim@yurim.skku.ac.kr