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COMPARISON EXAMPLES ON GENERALIZED QUASI-VARIATIONAL INEQUALITIES

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ABSTRACT. The purpose of this paper is to provide two examples which prove that Cubiotti's theorem and Yao's one on the generalized quasi-variational inequality problem are independent of each other. In addition, we give another example which tells us that certain conditions are essential in Cubiotti's theorem and Yao's one.

1. Introduction

Let E be a topological vector space, E^* its topological dual space, and X a nonempty subset of E. Let us denote the usual pairing between E^* and E by $\langle w, x\rangle$ for $w \in E^*$ and $x \in E$. Given two multifunctions $K : X \to 2^X$ and $F : X \to 2^{E^*}$, the generalized quasi-variational inequality problem (in short, GQVIP) is to find vectors $\bar{x} \in X$ and $\bar{z} \in F\bar{x}$ such that

$$\bar{x} \in K\bar{x} \quad \text{and} \quad \langle \bar{z}, x - \bar{x}\rangle \geq 0 \quad \text{for all} \quad x \in K\bar{x}.$$

The seminal paper on GQVIP is Chan and Pang's one [1]. Chan and Pang's result is as follows.

THEOREM 1. *Let X be a nonempty compact convex subset of R^n. Let $K : X \to 2^X$ be a continuous multifunction with nonempty compact convex values, and let $F : X \to 2^{R^n}$ be an upper semicontinuous multifunction with nonempty compact convex values. Then GQVIP has a solution.*

Chan and Pang [1] restricted their discussion to the finite dimensional Euclidean space R^n. Shih and Tan [6], from this observation,

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attacked the same problem in an infinite dimensional Hausdorff locally convex space. In both [1] and [6], all the existence results on GQVIP are obtained under some suitable continuity assumptions for the mappings K and F. Recently, Yao [7] sharpened Theorem 1 by weakening continuity assumptions on K and F as follows.

THEOREM 2 [7, Theorem 3.3]. *Let X be a nonempty compact convex subset of R^n, $K : X \to 2^X$ and $F : X \to 2^{R^n}$. Suppose that the following conditions are satisfied:*
(2.1) *for each $x \in X$, Fx is nonempty compact convex, and for every fixed $y \in X$, the mapping*

$$x \mapsto \inf_{z \in Fx} \langle z, x - y \rangle$$

is lower semicontinuous on X;
(2.2) *K has nonempty closed convex values and for every fixed $p \in R^n$, the following set is closed:*

$$\Big\{ x \in X \mid \langle p, x \rangle \le \sup_{y \in Kx} \langle p, y \rangle \Big\};$$

(2.3) *the interacting set $\{x \in X \mid \sup_{y \in Kx} \inf_{z \in Fx} \langle z, x - y \rangle \le 0\}$ is closed.*

Then GQVIP has a solution.

On the other hand, in a series of recent papers [2-5], Cubiotti obtained some nice generalizations of Theorem 1 by adopting a completely different approach from Yao's [7]. One of them is as follows.

THEOREM 3 [5, Theorem 3.2]. *Let X be a nonempty compact convex subset of R^n, $K : X \to 2^X$ and $F : X \to 2^{R^n}$. Suppose that the following conditions are satisfied:*
(3.1) *Fx is nonempty compact for each $x \in X$, and convex for each $x \in X$ with $x \in Fx$;*
(3.2) *for each $w \in X - X$, the set $\{x \in X \mid \inf_{z \in Fx} \langle z, w \rangle \le 0\}$ is closed;*
(3.3) *the multifunction K is lower semicontinuous with nonempty convex values;*

(3.4) *the set* $\{x \in X \mid x \in Kx\}$ *is closed.*

Then GQVIP has a solution.

As seen in the above, it seems that the conditions (2.1)~(2.4) in Theorem 2 and those (3.1)~(3.4) in Theorem 3 are very different. As a matter of fact, Yao and Guo [8, p.375] pointed out that the following weaker hypothesis than (2.1) does not imply (3.2):

$$\text{For each } y \in X, \text{ the set } \left\{x \in X \mid \inf_{z \in Fx} \langle z, x - y\rangle \leq 0\right\} \text{ is closed.}$$

The purpose of this paper is to provide some examples in multivalued settings which show that Theorems 2 and 3 are independent of each other. It will be shown that (3.2) does not imply (2.1). In addition, we give an example which tells us that the conditions (2.3) and (3.3) are essential in Theorems 2 and 3.

2. Examples

To begin with, let us recall some definitions. A *multifunction* $F : X \to 2^Y$ is a function from X into 2^Y the family of all nonempty subsets of Y. For topological spaces X and Y, a multifunction $F : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) provided that for each open subset V of Y, we have $\{x \in X \mid \emptyset \neq Fx \subset V\}$ is open in X; and *lower semicontinuous* (l.s.c.) provided that for each open subset V of Y, we have $\{x \in X \mid Fx \cap V \neq \emptyset\}$ is open in X.

The first example shows that Theorem 2 does not imply Theorem 3 in general.

EXAMPLE 1. Define $K : [0,1] \to 2^{[0,1]}$ by

$$Kx = \begin{cases} [1/2, 1] & \text{if} \quad x = 0 \\ 1/4 & \text{if} \quad 0 < x \leq 1, \end{cases}$$

and define $F : [0,1] \to 2^R$ by

$$Fx = \begin{cases} [1/2, 1] & \text{if} \quad x = 0 \\ 1/3 & \text{if} \quad 0 < x \leq 1. \end{cases}$$

(1) For each $x \in X$, Fx is nonempty compact convex, and for every fixed $y \in X$, the mapping

$$x \mapsto \inf_{z \in Fx} z(x-y)$$

is lower semicontinuous on X; Indeed,

$$gx = \inf_{z \in Fx} z(x-y) = \begin{cases} -y & \text{if} \quad x = 0 \\ \frac{1}{4}(x-y) & \text{if} \quad 0 < x \leq 1. \end{cases}$$

So the mapping $gx = \inf_{z \in Fx} z(x-y)$ is lower semicontinuous on $[0,1]$.
(2) K has nonempty closed convex values and for every fixed $p \in R$, the set $S = \{x \in X \mid px \leq \sup_{y \in Kx} py\}$ is closed. In fact,
Case (i) $p = 0$. It is obvious that $S = [0,1]$
Case (ii) $p > 0$.

$$\begin{aligned} S = \left\{x \in [0,1] \mid px \leq \sup_{y \in Kx} py\right\} &= \{0\} \cup \left\{x \in (0,1] \mid px \leq \sup_{y \in Kx} py\right\} \\ &= \{0\} \cup \left\{x \in (0,1] \mid px \leq \frac{1}{4}p\right\} \\ &= \{0\} \cup (0, 1/4] = [0, 1/4]. \end{aligned}$$

Case (iii) $p < 0$. $S = \left\{0 < x \leq 1 \mid px \leq \frac{1}{4}p\right\} = \{0 < x \leq 1 \mid x \geq 1/4\} = [1/4, 1]$.
(3) The interacting set $\{x \in X \mid \sup_{y \in Kx} \inf_{z \in Fx} z(x-y) \leq 0\}$ is closed. Indeed,

$$\begin{aligned} &\left\{x \in [0,1] \mid \sup_{y \in Kx} \inf_{z \in Fx} z(x-y) \leq 0\right\} \\ &= \{0\} \cup \left\{x \in (0,1] \mid \ \frac{1}{3}(x - \frac{1}{4}) \leq 0\right\} \\ &= \{0\} \cup (0, 1/4] = [0, 1/4]. \end{aligned}$$

Therefore by Theorem 2, there exist $\bar{x} \in [0,1]$ and $\bar{z} \in F\bar{x}$ such that $\bar{x} \in K\bar{x}$ and $\bar{z}(x - \bar{x}) \geq 0$ for all $x \in K\bar{x}$. In this case, $\bar{x} = 1/4$ and $\bar{z} = 1/3$ uniquely satisfy the above relation.

REMARK 1. Since F and K are neither u.s.c. nor l.s.c., we can not apply Theorem 3 to Example 1. Thus, Example 1 shows that Theorem 2 is not contained in Theorem 3.

The second example says that Theorem 3 does not imply Theorem 2 in general.

EXAMPLE 2. Define $K : [0,1] \to 2^{[0,1]}$ by

$$Kx = \begin{cases} [1/3, 3/4] & \text{if } \ 0 \leq x < \frac{1}{2} \\ 1/2 & \text{if } \ x \geq \frac{1}{2}, \end{cases}$$

and define $F : [0,1] \to 2^R$ by

$$Fx = \begin{cases} [-1,1] & \text{if } \ x = \frac{1}{2} \\ [-1,2] & \text{if } \ x \neq \frac{1}{2}. \end{cases}$$

(1) For each $x \in X$, Fx is nonempty compact convex.
(2) For each $w \in [0,1] - [0,1] = [-1,1]$, the set $\{x \in X \mid \inf_{z \in Fx} zw \leq 0\}$ is closed;
Case (i) $-1 \leq w < 0$.

$$\begin{aligned} \left\{x \in [0,1] \mid \inf_{z \in Fx} zw \leq 0\right\} &= \{1/2\} \cup \left\{x \neq \frac{1}{2} \mid \inf_{z \in Fx} zw \leq 0\right\} \\ &= \{1/2\} \cup \left\{x \neq \frac{1}{2} \mid \inf_{-1 \leq z \leq 2} zw \leq 0\right\} \\ &= \{1/2\} \cup \left\{x \neq \frac{1}{2} \mid 2w \leq 0\right\} \\ &= \{1/2\} \cup \left\{x \neq \frac{1}{2} \mid x \in [0,1]\right\} = [0,1]. \end{aligned}$$

Case (ii) $w = 0$. It is obvious that $\{x \in [0,1] \mid \inf_{z \in Fx} zw \leq 0\} = [0,1]$.
Case (iii) $0 < w \leq 1$. Since $\inf_{z \in Fx} zw = -w < 0$ for all $x \in [0,1]$, we have $\{x \in [0,1] \mid \inf_{z \in Fx} zw \leq 0\} = [0,1]$.
(3) Clearly K is lower semicontinuous with nonempty convex values.
(4) The set $\{x \in [0,1] \mid x \in Kx\} = [1/3, 1/2]$ is closed. Therefore by Theorem 3, there exist $\bar{x} \in [0,1]$ and $\bar{z} \in F\bar{x}$ such that $\bar{x} \in K\bar{x}$ and $\bar{z}(x - \bar{x}) \geq 0$ for all $x \in K\bar{x}$. For instance, $\bar{x} = 1/2$ and $\bar{z} = 1/2$ satisfy the above relation.

REMARK 2. In Example 2, it can be directly shown that for $y = 1$, the mapping

$$x \mapsto \inf_{z\in Fx} z(x-y) = \inf_{z\in Fx} z(x-1)$$

is not lower semicontinuous on X; In fact,

$$\inf_{z\in Fx} z(x-1) = \begin{cases} -\frac{1}{2} & \text{if } x = \frac{1}{2} \\ 2(x-1) & \text{if } x \neq \frac{1}{2}. \end{cases}$$

So the mapping $x \mapsto \inf_{z\in Fx} z(x-1)$ is not lower semicontinuous on $[0,1]$. Thus Theorem 2 is not applicable to Example 2. Therefore Example 2 tells us that Theorem 3 is not contained in Theorem 2 even if Kx is assumed to be compact convex for all $x \in [0,1]$, and F to be lower semicontinuous. Moreover Example 2 serves as an example showing that (3.2) does not imply (2.1).

The third example shows that the conditions (2.3) and (3.3) are essential in Theorems 2 and 3.

EXAMPLE 3. Define $K : [0,1] \to 2^{[0,1]}$ by

$$Kx = \begin{cases} \{1/2\} & \text{if } x \neq \frac{1}{2} \\ [0,1] & \text{if } x = \frac{1}{2}, \end{cases}$$

and define $F : [0,1] \to 2^R$ by

$$Fx = \begin{cases} [1/2, 1] & \text{if } x = 0 \\ 1/3 & \text{if } 0 < x \leq 1. \end{cases}$$

The unique fixed point $\bar{x}$ of K is $1/2$, and $\bar{z} = 1/3$, hence GQVIP has no solution in this case. It can be readily verified that K satisfies all the conditions of Theorem 2 but (2.3), and does those of Theorem 3 except (3.3). In fact, the auxiliary set $\{x \in [0,1] \mid \sup_{y\in Kx} \inf_{z\in Fx} z(x-y) \leq 0\} = [0,1/2)$ is not closed, and K is not lower semicontinuous. This implies that the conditions (2.3) and (3.3) can not be removable from the statements of Theorems 2 and 3 respectively.

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