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# ON THE COUNTABLE COMPACTA AND EXPANSIVE HOMEOMORPHISMS

In-Su Kim, Hisao Kato and Jong-Jin Park

Abstract. In this paper we introduce the notions of expansive homeomorphism and its properties, and study the relation between countable compacta and ordinal numbers. Our results extend and improve those of T. Kimura and others.

## 1. Introduction

In 1955, W. R. Utz [11] introduced the notion of "unstable homeomorphism"(=expansive homeomorphism) and studied its dynamical properties. Since then, it has been extensively studied in the area of topological dynamics, ergodic theory and continuum theory.

In [1], N. Aoki proved that every group automorphism of the Cantor set $C$ has P.O.T.P. M. Sears [10] proved that $\mathcal{E}(C)$ is dense in $\mathcal{H}(C)$, constructing a dense subset $\mathcal{A}$ of $\mathcal{E}(C)$ in $\mathcal{H}(C)$. M. Dateyama [3] proved that $\mathcal{P}(C)$ is dense in $\mathcal{H}(C)$, constructing a dense subset $\mathcal{B}$ of $\mathcal{P}(C)$ in $\mathcal{H}(C)$. B. F. Bryant [2], J. F. Jakobsen and W. R. Utz [5] proved that unit interval has no expansive homeomorphism. And T. Kimura [6] proved that the set of all expansive homeomorphisms with P.O.T.P. of the Cantor set C is dense in $\mathcal{H}(C)$ and he studied the existence of expansive homeomorphisms of a special countable compactum. Also he gave an example of an uncountable 0-dimensional compactum which admits no

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expansive homeomorphism. A space is 0-dimensional if it has a base consisting of closed and open sets.

Here, a question arises naturally as to what relation is between countable compacta and ordinal numbers. For the question, we shall give following results;

THEOREM 3.1. *Let $X$ and $Y$ be countable compacta. If $d(X) = d(Y) = \alpha$ and $X^{(\alpha)}$ is homeomorphic to $Y^{(\alpha)}$, then $X$ is homeomorphic to $Y$.*

THEOREM 3.2. *Let $X$ be a countable compactum with $d(X) = \alpha$. Then $X$ admits no expansive homeomorphism if $\alpha$ is a limit ordinal number.*

THEOREM 3.3. *Let $X$ be a countable compactum. If $d(X) = \alpha$ and $|X^{(\alpha)}| = 1$, then $\mathcal{E}(X) \cap \mathcal{P}(X) = \phi$.*

REMARK 3.4. There is a countable compactum $Y$ such that $d(Y) = 1, Y^{(1)} = \{a, b\}$, and $\mathcal{E}(Y) \cap \mathcal{P}(Y) \neq \phi$. Let $a_i (i \in \boldsymbol{Z})$ be different points of $\boldsymbol{E}^2$ such that $a_i \neq a_j (i \neq j)$, $\lim_{i \to \infty} a_i = a \neq \lim_{i \to -\infty} a_i = b$. Put $Y = \{a, b, a_i \mid i \in \boldsymbol{Z}\}$ and define a map $\varphi : Y \to Y$ by $\varphi(a) = a, \varphi(b) = b, \varphi(a_i) = a_{i+1}$. Then $\varphi \in \mathcal{E}(Y) \cap \mathcal{P}(Y)$.

## 2. Preliminaries

All spaces considered in this paper are assumed to be compact and metrizable space. A *compactum* is a compact and metrizable space. A *countable* compactum $X$ means that $X$ has a countable cardinality, i.e., $|X| \leq \infty$.

A homeomorphism $f$ of a compactum $X$ onto $X$ is said to be *expansive* [11] provided that there exists $c > 0$ such that if $x, y \in X$ with $x \neq y$, then there exists an integer $n \in \boldsymbol{Z}$ such that $d(f^n(x), f^n(y)) > c$, where $d$ is a metric of $X$. Then such a positive number $c > 0$ is called an *expansive constant* for $f$.

Let $f : X \to X$ be a homeomorphism of a compactum $X$. Given $\delta > 0$, a sequence $\{x_i \mid i \in \boldsymbol{Z}\}$ of points of $X$ is a *$\delta$-pseudo-orbit* of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for every $i \in \boldsymbol{Z}$. Given $\varepsilon > 0$, a sequence $\{x_i \mid i \in \boldsymbol{Z}\}$ of points of $X$ is $\varepsilon$ - *traced* by a point $y \in X$ if $d(f^i(y), x_i) < \varepsilon$ for every $i \in \boldsymbol{Z}$. We say that $f$ has the *pseudo-orbit tracing property* (abbrev. P.O.T.P.) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every $\delta$-pseudo-orbit of $f$ can be $\varepsilon$ -traced by some points of $X$.

For a compactum $(X, d)$, we denote by $\mathcal{H}(X)$ the space of all homeomorphisms of $X$ with the metric $\tilde{d}(f, g) = \max\{d(f(x), g(x)) \mid x \in X\}$ for $f, g \in \mathcal{H}(X)$. Let $\mathcal{E}(X) = \{f \in \mathcal{H}(X) \mid f \text{ is expansive}\}$ and $\mathcal{P}(X) = \{f \in \mathcal{H}(X) \mid f \text{ has P.O.T.P.}\}$.

The notation of the derived set of $X$ of order $\alpha$ can be found in [7]. The set $X^d$ of accumulation points of $X$ is called the *derived set* of $X$.

The derived set of $X$ of order $\alpha$ is defined by the conditions; $X^{(1)} = X^d$, $X^{(\alpha+1)} = (X^{(\alpha)})^d$ and $X^{(\lambda)} = \cap_{\alpha<\lambda} X^{(\alpha)}$ if $\lambda$ is a limit ordinal number. If $X^{(\alpha)} \neq \phi$ and $X^{(\alpha+1)} = \phi$, then we put $d(X) = \alpha$. And $d(X)$ is called the *derived degree* of $X$. It is well known that a compactum $X$ is a countable set if and only if $d(X)$ exists and it is a countable ordinal number. In this case, if $d(X) = \alpha$, then $X^{(\alpha)}$ is a finite set.

The notation of Hausdorff metric $H_d$ can be found in [9]. Let $X$ be a compactum with metric $d$. Consider

$$2^X = \{A \mid A \text{ is a nonempty closed subset of } X\}.$$

For each $\varepsilon > 0$ and each $A \in 2^X$, put

$$U(A;\varepsilon) = \{x \in X \mid d(x, a) < \varepsilon \text{ for some } a \in A\}$$

and

$$H_d(A, B) = \text{glb}\{\varepsilon > 0 \mid A \subset U(B;\varepsilon), B \subset U(A;\varepsilon)\}.$$

Then $2^X$ is a compactum with metric $H_d$, which is called the *Hausdorff metric* of $X$.

We consider following Lemma [6] to facilitate our discussion.

LEMMA 2.1 [6]. *Let $S = \{0, 1, 1/2, 1/3, \cdots\}$. Then*

- **(a):** *the set of all expansive homeomorphisms of $S$ is dense in $\mathcal{H}(S)$,*
- **(b):** *the set of all homeomorphisms with P.O.T.P. of $S$ is dense in $\mathcal{H}(S)$,*
- **(c):** *$S$ has no expansive homeomorphism with P.O.T.P.*

## 3. Main results on countable compacta

In this section, we assume that all ordinal numbers will mean countable ordinal numbers. The Euclidean 2-space (=the plane) is denoted by $\boldsymbol{E}^2$.

THEOREM 3.1. *Let $X$ and $Y$ be countable compacta. If $d(X) = d(Y) = \alpha$ and $X^{(\alpha)}$ is homeomorphic to $Y^{(\alpha)}$, then $X$ is homeomorphic to $Y$.*

*Proof.* First, by induction for any (countable) ordinal number $\alpha$ we will construct a countable compactum $X_\alpha$ in the plane $E^2$ such that $d(X_\alpha) = \alpha$ and $X_\alpha^{(\alpha)} = \{a\}$ is a one point set. Consider a sequence $\{x_i \mid x_i \in \boldsymbol{E}^2, i \in \boldsymbol{N}\}$, where $x_i \neq x_j (i \neq j)$, $x_i \neq a$ and $\lim_{i\to\infty} x_i = a (a \in \boldsymbol{E}^2)$. Let $X_1 = \{ x_i \mid i \in \boldsymbol{N}\} \cup \{a\}$. Assume that for each ordinal number $\beta < \alpha$, $X_\beta$ has been constructed. We consider two cases;

(1) $\alpha = \beta + 1$,

(2) $\alpha$ is a limit ordinal number.

**Case (1)** Consider a sequence $\{X_\beta^i \mid i \in \boldsymbol{N}\}$ of compacta of $\boldsymbol{E}^2$ such that $X_\beta^i$ is homeomorphic to $X_\beta$ , $X_\beta^i \cap X_\beta^j = \phi(i \neq j)$ and $\lim_{i\to\infty} H_d(X_\beta^i, \{a\}) = 0$, where $a \in \boldsymbol{E}^2$ and $a \notin X_\beta^i$. Let $X_\alpha = \cup_{i=1}^{\infty} X_\beta^i \cup \{a\}$.

**Case (2)** Assume $\alpha$ is a limit ordinal number. Choose a sequence $\{\alpha_i \mid i \in \boldsymbol{N}\}$ of ordinal numbers such that $\alpha_1 < \alpha_2 < \cdots < \alpha$ and $\lim_{i\to\infty} \alpha_i = \alpha$. By induction we can choose a sequence $\{X_{\alpha_i} \mid i \in \boldsymbol{N}\}$ of compacta of $\boldsymbol{E}^2$, where $d(X_{\alpha_i}) = \alpha_i$, $X_{\alpha_i} \cap X_{\alpha_j} = \phi(i \neq j)$ and $\lim_{i\to\infty} H_d(X_{\alpha_i}, \{a\}) = 0$, $a \notin X_{\alpha_i}$. Let $X_\alpha = \cup\{X_{\alpha_i} \mid i \in \boldsymbol{N}\} \cup \{a\}$. It is easily checked that $X_\alpha$ satisfies our desired conditions.

Second, by induction we will show that any countable compactum $X$ with $d(X) = \alpha$ and $X^{(\alpha)} = \{a\}$ is homeomorphic to $X_\alpha$. It is clear that any countable compactum $X$ with $d(X) = 1$ and $X^{(1)} = \{a\}$ is homeomorphic to $X_1$. For any ordinal number $\beta < \alpha$, we assume that any countable compactum $X$ with $d(X) = \beta$ and $X^{(\beta)} = \{a\}$ is homeomorphic to $X_\beta$. Let $X$ be a countable compactum with $d(X) = \alpha$ and $X^{(\alpha)} = \{a\}$ . Consider two cases:

(i) $\alpha$ is not a limit ordinal number.

(ii) $\alpha$ is a limit ordinal number.

**Case (i)** Note that $X^{(\alpha-1)}$ is homeomorphic to $X_1$. Put $X^{(\alpha-1)} = \{a, x_1, x_2, \cdots\}$. Then we can choose a sequence $\{Y_i \mid i \in \boldsymbol{N}\}$ of closed sets of $X$ such that $Y_i^{(\alpha-1)} = \{x_i\}, Y_i \cap Y_j = \phi (i \neq j)$, and $X = \cup_{i=1}^{\infty} Y_i \cup \{a\}$, $\lim_{i\to\infty} H_d(Y_i, \{a\}) = 0$. By induction, we see that $Y_i$ is homeomorphic to $X_{\alpha-1}$. Then we can easily construct a homeomorphism $\varphi : X_\alpha \to X$.

**Case (ii)** Note that for any open neighborhood $U$ of $a$ in $X$, $d(X - U) < \alpha$ and also note that for any ordinal number $\beta < \alpha$ there is a closed and open subset $A$ of $X$ such that $d(A) = \beta$ and $A^{(\beta)}$ is a one point set. Suppose that $\alpha_1 < \alpha_2 < \cdots$ is the sequence of ordinal numbers with $\lim_{i\to\infty} \alpha_i = \alpha$, where the sequence was chosen in the case of the construction of $X_\alpha$. By induction, we can choose a subsequence $\alpha_{n_1} < \alpha_{n_2} < \cdots$ of the sequence $\{\alpha_i \mid i \in \boldsymbol{N}\}$ and a sequence $\{Y_i \mid i \in \boldsymbol{N}\}$ of closed sets of $X$ such that $\cup_{i=1}^{\infty} Y_i \cup \{a\} = X$, $Y_i \cap Y_j = \phi (i \neq j)$, $\lim_{i\to\infty} H_d(Y_i, \{a\}) = 0$, $d(Y_i) = \alpha_{n_i}$ and $Y_i^{(\alpha_{n_i})}$ is a one point set. By induction, we see that $Y_i$ is homeomorphic to $\cup_{n_{i-1} < j \leq n_i} X_\alpha^j$ (see the above construction of $X_\alpha$). By using this fact, we can easily construct a homeomorphism $\varphi : X_\alpha \to X$.

Finally, we will show that if $X$ and $Y$ are countable compacta such that $d(X) = d(Y) = \alpha$ and $X^{(\alpha)} = \{a_1, \cdots, a_n\}$ is homeomorphic to $Y^{(\alpha)} = \{b_1, \cdots, b_n\}$, then $X$ is homeomorphic to $Y$. Choose closed sets $Z_i$ of $X$ and closed sets $Z'_i$ of $Y$ $(i = 1, \cdots, n)$ such that $Z_i \cap Z_j = \phi = Z'_i \cap Z'_j (i \neq j)$, $X = \cup\{Z_i \mid i = 1, \cdots, n\}$, $Y = \cup\{Z'_i \mid i = 1, \cdots, n\}, a_i \in Z_i$ and $b_i \in Z'_i$. Note that $d(Z_i) = d(Z_i') = \alpha$ and $Z_i^{(\alpha)}$, $Z_i'^{(\alpha)}$ are one point sets, respectively. By the above argument, we see that $Z_i$ is homeomorphic to $Z'_i$ for each $i$, and hence $X$ and $Y$ are homeomorphic. This completes the proof. □

THEOREM 3.2. *Let $X$ be a countable compactum with $d(X) = \alpha$. Then $X$ admits no expansive homeomorphism if $\alpha$ is a limit ordinal number.*

*Proof.* We will show that if $X$ admits an expansive homeomorphism $\varphi : X \to X$, then $d(X) = \alpha$ is not a limit ordinal number. Suppose, on the contrary, that $\alpha$ is a limit ordinal number. Put $X^{(\alpha)} = \{a_1, \cdots, a_n\}$. Let $c > 0$ be an expansive constant for $\varphi$. Since $\cap_{\beta<\alpha} X^{(\beta)} = X^{(\alpha)}$, there is an ordinal number $\beta < \alpha$ such that $X^{(\beta)} \subset U(\{a_1, \cdots, a_n\}; \frac{c}{3})$. Note that $\varphi(X^{(\beta)}) = X^{(\beta)}$. Since $\varphi$ is continuous, there is a sufficiently large ordinal number $\beta < \beta_0 < \alpha$ such that $\varphi(X^{(\beta_0)} \cap U(\{a_j\}; \frac{c}{3})) \subset X^{(\beta_0)} \cap U(\{\varphi(a_j)\}; \frac{c}{3})$, for all $j = 1, \cdots, n$. By induction we see that $\varphi^i(X^{(\beta_0)} \cap U(\{a_j\}; \frac{c}{3})) \subset X^{(\beta_0)} \cap U(\{\varphi^i(a_j)\}; \frac{c}{3})$, for all $j = 1, \cdots, n$ and $i \in \boldsymbol{Z}$. Choose $x, y \in (X^{(\beta_0)} \cap U(\{a_j\}; \frac{c}{3}))$ with $x \neq y$. Then $d(\varphi^i(x), \varphi^i(y)) < c$, for any $i \in \boldsymbol{Z}$. This is a contradiction. □

THEOREM 3.3. *Let $X$ be a countable compactum. If $d(X) = \alpha$ and $|X^{(\alpha)}| = 1$, then $\mathcal{E}(X) \cap \mathcal{P}(X) = \phi$.*

*Proof.* Suppose, on the contrary, that there is $\varphi \in \mathcal{E}(X) \cap \mathcal{P}(X)$. By Theorem 3.2, $\alpha$ is not a limit ordinal number. Then by Theorem 3.1, $X^{(\alpha-1)}$ is homeomorphic to $S$ in Lemma 2.1. Since $\varphi$ is a homeomorphism, $\varphi(X^{(\alpha-1)}) = X^{(\alpha-1)}$. Also, since $\varphi|X^{(\alpha-1)}$ is expansive, we can obtain a sequence $x_i (i \in \boldsymbol{Z})$ of points of $X^{(\alpha-1)}$ such that $\varphi(x_i) = x_{i+1}, \lim_{i\to\infty} x_i = \lim_{i\to-\infty} x_i = x_\infty$, where $\{x_\infty\} = X^{(\alpha)}$ (see (c) of the proof of Lemma 2.1). Put $\epsilon = (1/3) \cdot d(x_0, x_\infty) > 0$. Since $\varphi$ has P.O.T.P, there is $\delta > 0$ such that every $\delta$-pseudo-orbit is $\epsilon$-traced by some point of $X$. Choose a sufficiently large natural number $n \in \boldsymbol{N}$ such that $x_{-n}, x_n \in U(x_\infty; \delta)$.

Consider two finite sequences of points of $X$ as follows:

$$I_{n,0} = x_{-n}, x_{-n+1}, \cdots, x_n \ , \ I_{n,1} = \underbrace{x_\infty, x_\infty, \cdots, x_\infty}_{(2n+1)-times}.$$

Consider the following set

$$\mathcal{K} = \{\cdots, J_{-1}, J_0, J_1, J_2, \cdots \mid J_i = I_{n,0} \text{ or } I_{n,1}, \ i \in \boldsymbol{Z}\}$$

of $\delta$-pseudo-orbits of $\varphi$. Then $|\mathcal{K}| = 2^\omega$. By assumption, for $A \in \mathcal{K}$ there exists $x_A \in X$ such that $A$ is $\varepsilon$-traced by $x_A$. But if $A, A' \in \mathcal{K}$ and $A \neq A'$, then $x_A \neq x_{A'}$. Hence $|\{x_A | \ A \in \mathcal{K}\}| = |\mathcal{K}|$. Therefore $X$ is not countable. This is a contradiction. □

REMARK 3.4. There is a countable compactum $Y$ such that $d(Y) = 1$, $Y^{(1)} = \{a, b\}$, and $\mathcal{E}(Y) \cap \mathcal{P}(Y) \neq \phi$. Let $a_i (i \in \boldsymbol{Z})$ be different points of $\boldsymbol{E}^2$ such that $a_i \neq a_j (i \neq j)$, $\lim_{i \to \infty} a_i = a \neq \lim_{i \to -\infty} a_i = b$. Put $Y = \{a, b, a_i \mid i \in \boldsymbol{Z}\}$ and define a map $\varphi : Y \to Y$ by $\varphi(a) = a, \varphi(b) = b, \varphi(a_i) = a_{i+1}$. Then $\varphi \in \mathcal{E}(Y) \cap \mathcal{P}(Y)$.

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IN-SU KIM AND JONG-JIN PARK, DEPARTMENT OF MATHEMATICS, CHONBUK NATIONAL UNIVERSITY, CHONJU 561-756, KOREA
*E-mail*: insu@moak.chonbuk.ac.kr

HISAO KATO, INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA-SHI IBARAKI 305-8571, JAPAN