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THE JUMP NUMBER OF THE PRODUCT OF GENERALIZED CROWNS

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ABSTRACT. In this paper, we determine the jump number of the product of generalized crowns:

$$s(S_n^k \times S_m^l) = 2(m+l)(n+k) + 2(m-2)(n-2) - 1.$$

Let P be a finite ordered set and let $|P|$ be the number of elements of P. The *length* of a chain C in P is $|C| - 1$ and the *length* of P is the maximum length of a chain in P. A *bipartite* ordered set is an ordered set of length one. When $a < b$, we say that b *covers* a, denoted $a \prec b$, provided that for any $c \in P, a < c \leq b$ implies that $c = b$. A *linear extension* of P is a linearly ordered set L such that $a \leq b$ in L whenever $a \leq b$ in P. For a subset S of P, $\min(S)$ and $\max(S)$ denote the set of all minimal elements and all maximal elements, respectively, of S. For disjoint finite ordered sets P and Q, the *disjoint sum* $P + Q$ of P and Q is the ordered set on $P \cup Q$ such that $x < y$ if and only if $x < y$ in P or $x < y$ in Q, and the *linear sum* $P \oplus Q$ of P and Q is obtained from $P + Q$ by adding the new relations $x < y$ for all $x \in P$ and $y \in Q$.

For a linear extension L of a finite ordered set P, a (P, L)-*chain* is a maximal sequence of elements $z_1, z_2, \cdots, z_n$ such that $z_1 \prec z_2 \prec \cdots \prec z_n$ in both L and P. Let $c(L)$ be the number of (P, L)-chains. A consecutive pair (x_i, x_{i+1}) of elements in L is a *jump* of P in L if x_i is incomparable to x_{i+1} in P. The jumps induce a decomposition $L = C_0 \oplus C_1 \oplus \cdots \oplus C_m$ with (P, L)-chains $C_0, C_1, \cdots, C_m$, where $m = c(L)$ and $(\max(C_i), \min(C_{i+1}))$ is a jump of P in L for $i = 0, 1, \cdots, m-1$. Let $s(L, P)$ be the number of jumps of P in L. Then the *jump number* of P is defined to be the

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minimum number of $s(L,P)$ over all linear extensions L of P, denoted by $s(P)$. If $s(L,P) = s(P)$ then L is called an *optimal linear extension* of P. For a positive integer n, we denote by $\mathbf{n}$ the n-element chain.

In this paper we are concerned with the jump number of the product of certain ordered sets. We begin with two simple observations.

LEMMA 1. *Let P and Q be finite ordered sets and let $L = C_0 \oplus C_1 \oplus \cdots \oplus C_m$ be a linear extension of $P \times Q$ with $(P \times Q, L)$-chains $C_0, C_1, \cdots, C_m$. Then each C_i is of the form $\{x\} \times D$ or $E \times \{y\}$, where D and E are chains in Q and P, respectively.*

LEMMA 2. *For any finite ordered sets P and Q,*

$$s(P \times Q) \leq \sum_{i=0}^{r} \sum_{j=0}^{t} \min\{|C_i|, |D_j|\} - 1.$$

for any linear extensions $C_0 \oplus C_1 \oplus \cdots \oplus C_r$ and $D_0 \oplus D_1 \oplus \cdots \oplus D_t$ of P and Q, respectively.

EXAMPLE. There do not always exist such linear extensions of P and Q for which the equality in Lemma 2 holds. Let P and Q be the ordered sets in Figure 1.

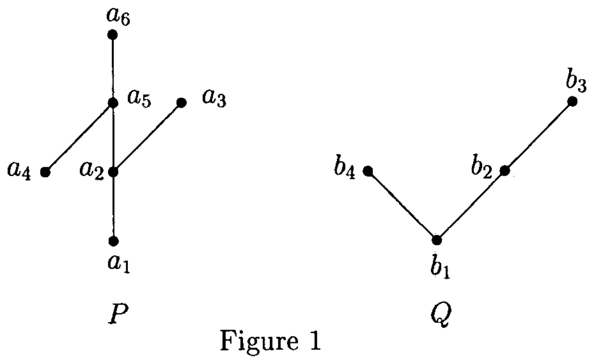


Figure 1

Then any linear extension $C_0 \oplus C_1 \oplus \cdots \oplus C_r$ of P is one of the following types:

$\mathbf{3} \oplus \mathbf{3}$, $\mathbf{1} \oplus \mathbf{4} \oplus \mathbf{1}$, $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{2}$, $\mathbf{2} \oplus \mathbf{3} \oplus \mathbf{1}$, $\mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{2}$, $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{2}$, $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}$, $\mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}$, $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{1} \oplus \mathbf{1}$, $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{1}$, and any linear extension $D_0 \oplus D_1 \oplus \cdots \oplus D_t$ of Q is one of the following types:

$$\mathbf{3} \oplus \mathbf{1},\ \mathbf{2} \oplus \mathbf{2},\ \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}.$$

In any case, we have $\sum_{i=0}^{r} \sum_{j=0}^{t} \min\{|C_i|, |D_j|\} - 1 \geq 7$. On the other hand, $s(P \times Q) = 6$. In fact, there is the following linear extension L of $P \times Q$ with $s(L, P \times Q) = 6$:

$$\begin{aligned} L = &\{(a_4, b_1), (a_4, b_2), (a_4, b_3)\} \oplus \{(a_1, b_1), (a_2, b_1), (a_5, b_1), (a_6, b_1)\} \\ &\oplus \{(a_1, b_2), (a_2, b_2), (a_5, b_2), (a_6, b_2)\} \oplus \{(a_1, b_3), (a_2, b_3), (a_5, b_3), (a_6, b_3)\} \\ &\oplus \{(a_3, b_1), (a_3, b_2), (a_3, b_3)\} \oplus \{(a_1, b_4), (a_2, b_4), (a_3, b_4)\} \\ &\oplus \{(a_4, b_4), (a_5, b_4), (a_6, b_4)\}, \end{aligned}$$

which implies that $s(P \times Q) \leq 6$. By Lemma 1, for any linear extension L of $P \times Q$, the length of the longest $(P \times Q, L)$-chain is 3 and L contains at most four $(P \times Q, L)$-chains with length 3. Since $| P \times Q |= 24$, we have $s(P \times Q) \geq 6$.

H. C. Jung [1] obtained the result that if T is an upward tree, that is, a finite ordered set containing no induced subset isomorphic to $(\mathbf{1} + \mathbf{1}) \oplus \mathbf{1}$, then there is an algorithm to find an optimal linear extension $C_0 \oplus C_1 \oplus \cdots \oplus C_r$ of T such that

$$s(T \times \mathbf{n}) = \sum_{i=0}^{r} \min\{|C_i|, n\} - 1.$$

For a linear extension L of a finite bipartite ordered set P, we denote the numbers of one-element (P, L)-chains and two-element (P, L)-chains by $o(P)$ and $t(P)$, respectively. Then the following is immediate from Lemma 2.

COROLLARY 3. *Let P and Q be finite bipartite ordered sets.*

$$s(P \times Q) \leq 2 \cdot t(P) \cdot t(Q) + t(P) \cdot o(Q) + o(P) \cdot t(Q) + o(P) \cdot o(Q) - 1$$

W. T. Trotter [2] defined the *generalized crown* S_n^k, for integers $n \geq 3$, $k \geq 0$, to be the bipartite ordered set with $\min(S_n^k) = \{a_1, a_2, \cdots, a_{n+k}\}$ and $\max(S_n^k) = \{b_1, b_2, \cdots, b_{n+k}\}$ such that each b_i is incomparable with

a_j for $j = i, i+1, \cdots, i+k$ and $b_i > a_j$ otherwise. We see that $o(S_n^k) = 2(n-2)$, $t(S_n^k) = k+2$ and $s(S_n^k) = 2n+k-3$ (see Figure 2 for S_4^2).

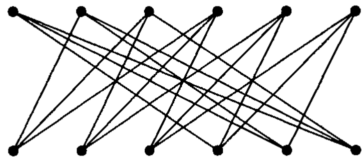


Figure 2. S_4^2

Now we determine the jump number of the product of generalized crowns in the following theorem.

THEOREM. For any integers n, k, m and l with $n, m \geq 3$ and $k, l \geq 0$,

$$s(S_n^k \times S_m^l) = 2(m+l)(n+k) + 2(m-2)(n-2) - 1.$$

Proof. Consider $H = S_n^k \times S_m^l$ for fixed integers $n \geq 3$, $k \geq 0$ and let

$$A = \min(S_n^k) = \{a_1, a_2, \cdots, a_{n+k}\}, \quad B = \max(S_n^k) = \{b_1, b_2, \cdots, b_{n+k}\},$$
$$C = \min(S_m^l) = \{c_1, c_2, \cdots, c_{m+l}\}, \quad D = \max(S_m^l) = \{d_1, d_2, \cdots, d_{m+l}\}.$$

Then we observe that $\min(H) = A \times C$ and $\max(H) = B \times D$. From Lemma 1, we first see that there do not exist $(a, c) \in A \times C$ and $(b, d) \in B \times D$ such that $(a, c) \prec (b, d)$.

Let L be an arbitrary linear extension of H. For all s, t with $1 \leq s \leq n+k$ and $1 \leq t \leq m+l$, let (a_s, d_s') and (b_t', c_t) be the least elements of $\{a_s\} \times D$ and $B \times \{c_t\}$, respectively, in L. Rearranging the subscripts, we may assume that $(a_1, d_1') < (a_2, d_2') < \cdots < (a_{n+k}, d_{n+k}')$ and $(b_1', c_1) < (b_2', c_2) < \cdots < (b_{m+l}', c_{m+l})$ in L. Suppose, without loss of generality, that $(a_{n-2}, d_{n-2}') < (b_{m-2}', c_{m-2})$ in L. Then $(a_{n-2}, d_{n-2}') < (b_1', c_1)$ or $(b_p', c_p) < (a_{n-2}, d_{n-2}') < (b_{p+1}', c_{p+1})$ for some p. Since the former case can be treated similarly, we only assume the latter. Set

$$S = \bigcup_{i=1}^{p} d(b_i', c_i) \text{ and } T = \bigcup_{j=1}^{n-2} d(a_j, d_j'),$$

where $d(x,y) = \{(a,c) \in A \times C \mid (a,c) < (x,y) \text{ in } H\}$. We shall prove that $S \cup T$ contains the at least $(m-2)(n-2)$ one-element (H,L)-chains. Since $S = \bigcup_{i=1}^{p} d(b'_i,\, c_i)$ is a disjoint union and $|d(b'_i,\, c_i)| = n-1$, it follows that $|S| = p(n-1)$. Similarly, we have $|T| = (n-2)(m-1)$.

CLAIM 1. For any element $(b,\, c_i) \in B \times \{c_i\}$ $(1 \le i \le p)$ with $b \ne b'_i$, if $(b,\, c_i)$ makes a two-element (H,L)-chain with an element of $S \cup T$, then there is a unique integer j with $1 \le j \le n-2$ such that $d(b'_i,\, c_i) \cap d(a_j,\, d'_j) = \emptyset$.

Let $(b,\, c_i)$ be any element of $B \times \{c_i\}$ with $(b, c_i) \ne (b'_i,\, c_i)$ for $1 \le i \le p$. Then $(b'_i,\, c_i) < (b,\, c_i)$ in L and $d(b, c_i) \cap d(b'_j,\, c_j) = \emptyset$ for all j with $j \ne i$. Hence there is no element $(x,y) \in S$ such that $[(x,y),(b,c_i)]$ is a two-element (H,L)-chain. Suppose that there is an element $(a,c_i) \in T-S$ such that $[(a,c_i),(b,c_i)]$ is a two-element (H,L)-chain. Then there is a unique integer j with $1 \le j \le n-2$ such that $a = a_j$ and $(a,\, c_i) \in d(a_j,\, d'_j)$. Since $d(b'_i,\, c_i) \cap d(a_j,\, d'_j) \subseteq \{(a_j,\, c_i)\}$ and $(a_j,\, c_i) \prec (b,\, c_i)$ in L, it follows that $(b'_i, c_i) < (a_j, c_i)$ in L, that is, (b'_i, c_i) is incomparable with (a_j, c_i) in H. Thus $d(b'_i,\, c_i) \cap d(a_j,\, d'_j) = \emptyset$.

CLAIM 2. For any element $(a_j,\, d) \in \{a_j\} \times D$ $(1 \le j \le n-2)$ with $d \ne d'_j$, if $(a_j,\, d)$ makes a two-element (H,L)-chain with an element of $S \cup T$, then there is a unique integer i with $1 \le i \le p$ such that $d(b'_i,\, c_i) \cap d(a_j,\, d'_j) = \emptyset$.

This claim can be proved by a similar method to Claim 1.

CLAIM 3. For $p+1 \le i \le m+l$, every element (b,c_i) of $B \times \{c_i\}$ cannot make a two-element (H,L)-chain with any element of $S \cup T$.

Suppose that, for $p+1 \le i \le m+l$, there are elements $(b,\, c_i) \in B \times \{c_i\}$ and $(a,\, c_i) \in S \cup T$ such that $[(a,\, c_i),(b,\, c_i)]$ is a two-element (H,L)-chain. But $(a,c_i) < (a_{n-2}, d'_{n-2}) < (b,c_i)$ in L, which is a contradiction.

CLAIM 4. For $n-1 \le j \le n+k$, every element (a_j,d) of $\{a_j\} \times D$ cannot make a two-element (H,L)-chain with any element of $S \cup T$.

This claim can be proved by a similar method to Claim 3.

Set $X = \{ (b'_i, c_i) \mid 1 \le i \le p \}$ and $Y = \{ (a_j, d'_j) \mid 1 \le j \le n-2 \}$. Let U be the subset of $[(A \times D) \cup (B \times C)] - (X \cup Y)$ each of whose elements makes a two-element (H, L)-chain with an element of $S \cup T$ and let $V = \{ (i, j) \mid 1 \le i \le p \text{ and } 1 \le j \le n-2 \text{ with } d(b'_i, c_i) \cap d(a_j, d'_j) = \emptyset \}$.

Now we define a map ψ from U to V. For $u \in U$, Claims 3 and 4 imply that $u \in B \times \{c_i\}$ for some i with $1 \le i \le p$ or $u \in \{a_j\} \times D$ for some j with $1 \le j \le n-2$. In either case, there is a unique $(a_j, c_i) \in A \times C$ such that $(a_j, c_i) \prec u$ in L with $1 \le i \le p$ and $1 \le j \le n-2$. Set $\psi(u) = (i, j)$. Since Claims 1 and 2 imply that $d(b'_i, c_i) \cap d(a_j, d'_j) = \emptyset$, ψ is a well-defined 1-1 map from U to V, whence $|U| \le |V|$. Since $|S \cap T| = p(n-2) - |V|$, we have

$$\begin{aligned} |S \cup T| &= |S| + |T| - |S \cap T| \\ &= p(n-1) + (n-2)(m-1) - p(n-2) + |V| \\ &= (n-2)(m-2) + (n-2) + p + |V| \\ &\ge (n-2)(m-2) + |X| + |Y| + |U|. \end{aligned}$$

Now we conclude that $|S \cup T|$ has at least $(n-2)(m-2)$ elements except for the elements which can make two-element (H, L)-chains, that is, there are at least $(n-2)(m-2)$ elements in $A \times C$ which are one-element (H, L)-chains.

Note that $o(S_n^k) = 2(n-2)$, $t(S_n^k) = k+2$, $o(S_m^l) = 2(m-2)$ and $t(S_m^l) = l+2$. By Corollary 3, we have

$$s(H) \le 2(m+l)(n+k) + 2(m-2)(n-2) - 1.$$

To prove the other inequality, let L be any linear extension of H. By the above argument L has at least $(m-2)(n-2)$ one-element (H, L)-chains in $A \times C$. Since H is self-dual, L also has at least $(m-2)(n-2)$ one-element (H, L)-chains in $B \times D$. Thus there are at most $2(n+k)(m+l) - 2(m-2)(n-2)$ two-element (H, L)-chains in H. Hence we have

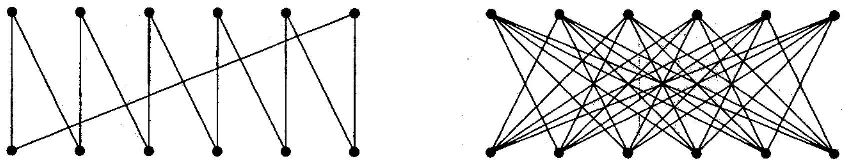
$$\begin{aligned} s(H, L) &\ge 2(n+k)(m+l) - 2(m-2)(n-2) + 4(m-2)(n-2) - 1 \\ &= 2(m+l)(n+k) + 2(m-2)(n-2) - 1, \end{aligned}$$

and so

$$s(H) \ge 2(m+l)(n+k) + 2(m-2)(n-2) - 1,$$

as desired. □

For $n \geq 3$, the *crowns* $C_n = S_3^{n-3}$ (see Figure 3 for C_6) and the *n-dimensional standard ordered sets* $S_n = S_n^0$ (see Figure 3 for S_6) are important examples in theory of ordered sets. From our theorem we have that $s(C_n \times C_m) = 2nm + 1$ and that $s(S_n \times S_m) = 4(n-1)(m-1) + 3$.



C_6 S_6

Figure 3

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