LOCAL STRUCTURE OF TRAJECTORY
FOR EXTREMAL FUNCTIONS

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ABSTRACT. In this note we study more about the omitted arc for
the extremal functions and its $\frac{\pi}{4}$-property based upon Schiffer's vari-
ational method and Brickman-Wilken's result. We give an example
other than the Koebe function which is both a support point of $S$
and the extreme point of $HS$. Furthermore, we discuss the relations
between the support points and the Löwner chain.

1. Introduction

Let $\Delta$ be the open unit disk in the complex plane $\mathbb{C}$, and let $H(\Delta)$
denote the linear space of holomorphic functions in $\Delta$, endowed with
the usual topology of local uniform convergence. A particular subset of
$H(\Delta)$ is the class $S$ which consists of all functions $f$ which are univalent
in $\Delta$ and normalized so that $f(0) = 0$ and $f'(0) = 1$.

For the study of linear extremal problems in $S$ it is natural to consider
two sets of functions, the support points of $S$ and the extreme points
of $S$.

We call $f \in S$ a support point of $S$ if there exists a continuous linear
functional $J$ defined on $H(\Delta)$ which is non-constant on $S$ and

$$ReJ(f) = \max_{g \in S} ReJ(g).$$

$f \in S$ is an extreme point of $S$ provided for $0 < t < 1$, $g \in S$, $h \in S,$

$$f = tg + (1-t)h \text{ implies that } f = g = h.$$

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It is well known that all rotations of the Koebe functions

\[ k_\theta(z) = \frac{z}{(1 - e^{i\theta} z)^2} \]

maps \( \Delta \) onto the complement of a ray from \(-\frac{1}{4}e^{-i\theta}\) to \(\infty\). A single slit mapping is a slit mapping whose range is the complement of a single Jordan arc. Functions in \( S \) that map \( \Delta \) onto the complement of a single Jordan arc are known to play a crucial role in the study of extremal problems for \( S \).

Schiffer ([14]) showed that for a quite general functional \( J \), any solution to \( \{ \max \text{Re}J(g) : g \in S \} \) maps \( \Delta \) onto the complement of a finite number of analytic Jordan arcs, and he determined a differential equation for the arcs in terms of parameters involving the extremal functions.

Goluzin ([7]) showed that if \( J \) has the special form

\[ J(g) = \sum_{i=1}^{n} b_i g^{(i)}(0), \quad (n \geq 2), \]

then \( \mathbb{C} \setminus f(\Delta) \) consists of finitely many arcs with the \( \frac{\pi}{4} \)-property; that is, the angle between the position vector and the tangent vector at any point on the slit is smaller in magnitude than \( \frac{\pi}{4} \).

For the particular functional \( J(f) = \text{Re} \ a_n \), where \( f(z) = z + \sum_{k=1}^{\infty} a_k z_k \), Schiffer ([15]) verified that \( J(f^2) \neq 0 \) and any extremal function maps \( \Delta \) onto the complement of a single analytic slit with an asymptotic direction at \( \infty \) and this slit possesses the \( \frac{\pi}{4} \)-property.

Pfluger ([13]) generalized this result by showing that any extremal function for

\[ \max_{g \in S} \text{Re}J(f) \quad (J \text{ non-constant on } S) \]

maps \( \Delta \) onto the complement of an analytic slit which has the \( \frac{\pi}{4} \)-property and an asymptotic direction at \( \infty \). Brickman and Wilken ([3]) found a considerably simpler proof of this result.

In this note we study more about the omitted arc and \( \frac{\pi}{4} \)-property based upon Schiffer's variational method and Brickman and Wilken's result. We provide local structure of trajectories that the range of an
extremal function is the complement of a Jordan analytic arc satisfying a certain differential equation. Furthermore, we discuss the relations between the support points and the Löwner chain.

2. Local Structure of Trajectories

According to the Schiffer's result, if $\Gamma$ is the complement of the range of an extremal function, $\Gamma$ consists of a collection of analytic arcs satisfying a differential equation of the form $Q(w)dw^2 > 0$, where $Q$ is analytic on $\Gamma$. Such an expression $Q(w)dw^2$ is called a quadratic differential and the arcs for which $Q(w)dw^2 > 0$ are called its trajectories. The following Schiffer's variational method will give us much more precise information about the omitted arc.

**Lemma 1 (Schiffer).** Let $J$ be a continuous functional on $H(\Delta)$, and let $f \in S$ be a point where $\text{Re}\{J\}$ attains its maximum value on $S$. Suppose that $J$ has a Fréchet differential $l(\cdot; f)$ which is not constant on $S$. Then $f$ maps the unit disk $\Delta$ onto the complement of a system of finitely many analytic arcs $w = w(t)$ satisfying the differential equation

$$\frac{1}{w^2} l \left( \frac{f^2}{f - w}; f \right), \left( \frac{dw}{dt} \right)^2 > 0.$$ 

**Lemma 2 (Brickman and Wilken).** Each extreme point of $S$ and each support point of $S$ have the monotonic modulus property, i.e., it maps $\Delta$ onto the complement of an arc which extends to $\infty$ with increasing modulus.

**Theorem 2.1 (Duren [5]).** Let $J$ be a continuous linear functional on $H(\Delta)$ which is not constant on $S$ and let $f$ maximize $\text{Re}\{J\}$ on $S$. Then $f$ maps $\Delta$ onto the complement of a single analytic arc $\Gamma$ which satisfies the differential equation

$$(2.1) \quad \frac{1}{w^2} J \left( \frac{f^2}{f - w} \right) dw^2 > 0.$$ 

At each point $w \in \Gamma$ except perhaps the finite tip, the tangent line makes angle of less than $\frac{\pi}{4}$ with the radical line from 0 to $w$. 

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Proof. Since a continuous linear functional is its own Fréchet differential, Lemma 1 shows that $\Gamma$ consists of finitely many analytic arcs satisfying the differential equation (2.1). In fact, $\Gamma$ is a single unbranched arc which extends to $\infty$ with monotonic modulus by Lemma 2.

Choose a point $w \in \Gamma$, not an endpoint, and consider the function

$$g = \frac{wf}{w - f}.$$

Observe that $g$ belongs to $S$ and maps $\Delta$ onto the complement of two disjoint arcs extending to $\infty$. Thus $g$ is not a support point, and so

$$(2.2) \quad Re\{J(g)\} < Re\{J(f)\}.$$

Since $J$ is linear, (2.2) is equivalent to

$$(2.3) \quad Re \left\{ J \left( \frac{f^2}{f - w} \right) \right\} > 0, \quad w \in \Gamma,$$

where $w$ is not an endpoint of $\Gamma$.

The inequality (2.3) has two consequences. First, the fact that $J(\frac{f^2}{f - w}) \neq 0$ assures that the quadratic differential has no singularities on $\Gamma$, except perhaps at the endpoints, so that $\Gamma$ has no corners. In other words, $\Gamma$ is a single analytic arc. Second, the inequality (2.3) may be combined with (2.1) to show that

$$Re \left\{ \left( \frac{dw}{w} \right)^2 \right\} > 0 \quad \text{on} \quad \Gamma,$$

which is equivalent to $|arg\{\frac{dw}{w}\}| \leq \frac{\pi}{4}$. This completes the proof. \hfill \Box

Remark. A recent result of Duren, Leung and Schiffer ([6]) shows that under very general conditions the omitted arc is a half line whenever it has a radial angle of $\pm \frac{\pi}{4}$ at its tip.
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**Lemma 3** (Schiffer). Let \( J \) be a continuous linear functional on \( H(\Delta) \) which is not constant on \( S \), and let \( f \) maximize \( \text{Re}\{J\} \) on \( S \). Then

\[
J(f^2) \neq 0.
\]

**Theorem 2.2** (Duren [5]). Let \( J \) be a continuous linear functional on \( H(\Delta) \) which is not constant on \( S \), and let \( f \) maximize \( \text{Re}\{J\} \) on \( S \). Then the arc \( \Gamma \) omitted by \( f \) is asymptotic to the half-line

\[
w = \frac{1}{3} \frac{J(f^3)}{J(f^2)} - J(f^2)t, \quad t \geq 0,
\]

at \( \infty \). Furthermore, the radial angle \( \text{arg} \left\{ \frac{dw}{w} \right\} \) of \( \Gamma \) tends to 0 at \( \infty \).

**Proof.** Let \( \Gamma \) be parametrized by \( w = w(t), \ 0 < t < \infty \), in such a way that \( w(t) \to \infty \) as \( t \to 0 \) and the differential equation (2.1) takes the form

\[
\frac{1}{w^2} J \left( \frac{f^2}{f - w} \right) \left( \frac{dw}{dt} \right)^2 = 1.
\]

By Lemma 3, we have \( J(f^2) \neq 0 \). So the substitution \( w = u^{-2} \) transforms \( \Gamma \) to an analytic curve

\[u = b_1 t + b_3 t^3 + \cdots\]

through the origin which satisfies

\[
-4J \left( \frac{f^2}{1 - fu^2} \right) \left( \frac{du}{dt} \right)^2 = 1,
\]

or

\[
(c_0 + c_1 b_1 t^2 + \cdots)(b_1^2 + 6b_1 b_3 t^2 + \cdots) = -\frac{1}{4},
\]

where \( c_n = J(f^{n+2}), \ n = 0, 1, 2, \ldots \). Equating coefficients, we obtain

\[
(2.5) \quad c_0 b_1^2 = -\frac{1}{4}, \quad c_1 b_1^4 + 6c_0 b_1 b_3 = 0.
\]
On the other hand,

\[ w = u^{-2} = b_1^{-2}t^{-2} - 2b_1^{-3}b_3 + O(t^2), \quad t \to 0. \]

Thus \( \Gamma \) is asymptotic to the line

\[ w = \alpha + \beta t, \quad t \to \infty, \]

where \( \alpha = -2b_1^{-3}b_3 \) and \( \beta = b_1^{-2} \). But the equations (2.5) give

\[ b_1^{-2} = -4c_0, \quad b_1^{-3}b_3 = -\frac{c_1}{6c_0}. \]

This proves that \( \Gamma \) approaches the half-line (2.4) near \( \infty \). In particular, \( \arg w \to \arg\{-J(f^2)\} \) as \( w \to \infty \) along \( \Gamma \).

Since \( J\left(\frac{f^2}{f-w}\right) = -\frac{J(f^2)}{w} + O\left(\frac{1}{w^2}\right) \), it follows that \( \arg J\left(\frac{f^2}{f-w}\right) \to 0 \). Thus the differential equation

\[ J\left(\frac{f^2}{f-w}\right) \left(\frac{dw}{w}\right)^2 > 0 \]

shows that the radial angle \( \arg\{\frac{dw}{w}\} \to 0 \) as \( w \to \infty \) along \( \Gamma \). \( \square \)

**Remark.** Hengartner and Schober ([8]) used the monotone modulus property to show that for every support point \( f \in S \), both \( \frac{f(z)}{z} \) and \( \log \left[ \frac{f(z)}{z} \right] \) are univalent in \( \triangle \). The \( \frac{\pi}{4} \)-property was used in [9] to show that for every support point \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S, \quad |a_2| > 1 \) and \( |a_3| > \frac{3}{8} \).

Kirwan and Pell ([10]) improved these estimates to

\[ |a_2| > \sqrt{2} \quad \text{and} \quad |a_3| > 1, \]

and they produced an example for which

\[ |a_2| < 1.774. \]

The sharp lower bounds are unknown.
3. Examples

It is well known in [5] that the Koebe function

\[ k_x(z) = \frac{z}{(1-xz)^2}, \quad |x| = 1. \]

uniquely maximize \( ReJ_x \) over \( S \), where \( J_xg = \bar{x}g(0), \ |x| = 1. \)

Thus, the Koebe functions \( k_x \) are both support points of \( S \) and extreme points of \( HS \), the closed convex hull of \( S \).

Let \( C \) denote the subclass of \( S \) which consists of close-to-convex functions and let \( HC \) the closed convex hull of \( C \).

It is known in [2] that the functions

\[ f_{xy}(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}, \quad |x| = |y| = 1, \quad x \neq y, \]

are both support points of \( C \) and extreme points of \( HC \). If we set \( y = 1, \ x \neq 1 \) and \( a = \frac{1}{2}(1+x) \), the tip of omitted arc \( \Gamma \) is

\[ f_{xy}\left(\frac{1}{2a-1}\right) = -\frac{1}{4(1-a)} = -\frac{i}{4} \cot \frac{\theta}{2} \text{ when } x = e^{i\theta}, \ 0 < \theta < 2\pi. \]

Thus, with varying \( \theta \), the rays obtained consists of all rays through \( w = -\frac{1}{2} \) and having the tip on the line \( Rew = -\frac{1}{4} \). It is evident that if \( |\cot \frac{\theta}{2}| > 1 \), then the ray will not have strictly increasing modulus. Therefore, if \( |\theta| < \frac{\pi}{2} \), then \( C \setminus f_{xy}(\Delta) \) does not have strictly increasing modulus and thus \( f_{xy} \) is not a support point of \( S \).

However, if the omitted half-line \( \Gamma \) is oriented so that \( \Gamma \) is traversed from the tip \( P \) of \( \Gamma \) to \( \infty \), a computation shows that \( |arg(-\frac{x}{y})| \) is the angle between the tangent vector to \( \Gamma \) and the radius vector to \( \Gamma \) at \( P \). It is easily seen that the radial angle \( arg\left(\frac{dw}{w}\right) \) of \( \Gamma \) decreases monotonically to 0 as \( \Gamma \) traverses from \( P \) to \( \infty \).

Thus, if \( \frac{\pi}{4} < |arg(-\frac{x}{y})| < \pi \), then \( f_{xy} \) can be neither a support point of \( S \) nor an extreme point of \( HS \) because \( \Gamma \) fails to satisfy the \( \frac{\pi}{4} \)-property.
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If $|\arg(-\frac{x}{y})| < 0$, i.e., if $-x = y$, then $f_{xy}$ is the Koebe function $k_y(z) = \frac{x}{(1-yz)^2}$. If $0 < |\arg(-\frac{x}{y})| \leq \frac{\pi}{4}$, then $\Gamma$ does not violate the $\frac{\pi}{4}$-property. So if $0 < |\arg(-\frac{x}{y})| \leq \frac{\pi}{4}$, then the function

$$f_{xy}(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}$$

is both a support point of $S$ and an extreme point of $HS$.

4. The Support Points and the Löwner Chain

Suppose $f \in S$ maps $\triangle$ onto the complement of a Jordan arc which extends to $\infty$. By Löwner theory ([11]), $f$ may be embedded in a family of mappings

$$\{f(z,t)|0 \leq t < \infty\},$$

called a Löwner chain, with the following properties;

(i) $f(z,0) = f(z)$

(ii) $f(z,t_1)$ is subordinate to $f(z,t_2)$ if $t_1 < t_2$.

(iii) $e^{-t}f(z,t) \in S, \quad 0 \leq t < \infty$.

(iv) $\frac{\partial f(z,t)}{\partial t} = z\frac{\partial f(z,t)}{\partial z} \frac{1+\eta(t)z}{1-\eta(t)z}$, where $|\eta(t)| = 1, \quad z \in \triangle$.

From (ii) it follows that for any $t \geq 0$, $f(z) = f[\phi(z,t),t]$ where $\phi(z,t) \in H(\triangle)$, is one-to-one in $\triangle$ and satisfies $\phi(0,t) = 0$, $\phi'(0,t) = e^{-t}$.

$f(z,t)$ carries a point of the unit circle to the tip of the slit bounding $f(\triangle,t)$.

**Theorem 4.1.** If $f$ is an extreme point of $S$ and $f$ is embedded in the Löwner chain $\{f(z,t), t \geq 0\}$, then for all $t \geq 0$, $e^{-t}f(z,t)$ is an extreme point of $S$.

**Proof.** Suppose not, i.e., for some $t \geq 0$, suppose that

$$e^{-t}f(z,t) = sf_1(z) + (1-s)f_2(z), \quad (z \in \triangle)$$

where $0 < s < 1$, $f_1, f_2 \in S$ and $f_1 \neq f_2$. Then

$$f(z,t) = se^tf_1(z) + (1-s)e^tf_2(z).$$
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By the subordination property of Löwner chain,

\[ f(z) = f(z, 0) = f[\phi(z, t), t] = se^t f_1[\phi(z, t), t] + (1 - s)e^t f_2[\phi(z, t), t]. \]

Since \( e^t f_1[\phi(z, t), t] \) and \( e^t f_2[\phi(z, t), t] \) are distinct functions in \( S \), it contradicts the fact that \( f(z) \) is an extreme point of \( S \). \qed

**Lemma 4 (Duren [5]).** Each continuous linear functional \( J \) on the space \( H(\Delta) \) has the form

\[ J(f) = \int \int_E f(z) d\mu(z), \quad f \in H(\Delta), \]

where \( \mu \) is a complex-Borel measure supported on a compact subset \( E \) of \( \Delta \).

**Theorem 4.2.** Let \( J \) be a continuous linear functional on \( H(\Delta) \) which is not constant on \( S \), and let \( f \) maximize \( Re\{J\} \) on \( S \). If \( f \) is embedded in the Löwner chain \{\( f(z, t), \ t \geq 0 \)\}, then for all \( t \geq 0 \), \( e^{-t} f(z, t) \) is a support point of \( S \).

**Proof.** Let \( f \in S \) satisfy

\[ ReJ(f) = \max_{g \in S} ReJ(g) \]

where \( J \) is non-constant on \( S \). By Lemma 4, we can write

\[ J(f) = \int \int_E f(z) d\mu(z) \]

where \( \mu \) is a complex-Borel measure supported on a compact subset \( E \) of \( \Delta \). Denote by \{\( f(z, t) \)\} the Löwner chain associated with \( f \). Then by the subordination property of Löwner chain,

\[ J(f) = \int \int_E f(z) d\mu(z) = \int \int_E f(z, 0) d\mu(z) = \int \int_E f[\phi(z, t), t] d\mu(z). \]

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Setting \( \zeta = \phi(z, t) \) we get \( z = \phi^{-1}(\zeta, t) \). Thus,

\[
J(f) = \int \int_{\phi(E, t)} f(\zeta, t) d\mu[\phi^{-1}(\zeta, t)] \\
= \int \int_{\phi(E, t)} e^{-t} f(\zeta, t) e^{t} d\mu[\phi^{-1}(\zeta, t)].
\]

(4.1)

Since \( dv(\zeta, t) = e^{t} d\mu[\phi^{-1}(\zeta, t)] \) is a complex Borel measure supported on the compact set \( \phi(E, t) \subset \Delta \), we may define a continuous functional \( J_t \) such that

\[
J_t(g) = \int \int_{\phi(E, t)} g(\zeta) dv(\zeta, t).
\]

It follows from (4.1) that

\[
J(f) = J_t[e^{-t} f(z, t)].
\]

(4.2)

Moreover, if \( g \in S \), then by another change of variable we see that

\[
J_t(g) = J\{e^{t} g[\phi(z, t)]\}
\]

(4.3)

and \( e^{t} g[\phi(z, t)] \in S \). From (4.2) and (4.3) we obtain

\[
Re J_t[e^{-t} f(z, t)] = \max_{g \in S} Re J_t(g).
\]

Since \( e^{t} \phi(z, t) \) is a bounded univalent function, \( e^{t} \phi(z, t) \) is not a support point of \( S \). So

\[
Re J_t(f) = Re J[e^{t} f(z, t)] < Re J(f) = Re J_t[e^{-t} f(z, t)]
\]

and we conclude that \( J_t \) is non-constant on \( S \). This completes the proof. \( \square \)
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References


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