Variational-Type Inequalities on Reflexive Banach Spaces

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In this paper, we consider the existence of solutions to the variational-type inequalities for single-valued mappings and set-valued mappings on reflexive Banach spaces using Fan's section theorem.

1. Introduction and preliminaries

Variational inequalities introduced by Hartman and Stampacchia[5] have been extended and generalized in various directions as a powerful tool of current mathematical technology.


In this paper, we consider the existence of the solutions to the variational-type inequalities for single-valued mappings on reflexive Banach spaces, under different conditions from Behera and Panda[3]. And we consider the existence of the solutions to the variational-type inequalities for set-valued mappings on reflexive Banach spaces.

Now we introduce the following famous Fan's section theorem[4].

Theorem 1.1. Let \( K \) be a nonempty compact convex subset of a Hausdorff topological vector space \( X \). Let \( A \) be a subset of \( K \times K \) satisfying the following conditions;

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(1) for each \( x \in K \), \( (x, x) \in A \),
(2) for each fixed \( x \in K \), the set \( A_x := \{ y \in K : (x, y) \in A \} \) is closed in \( K \),
(3) for each fixed \( y \in K \), the set \( A^y := \{ x \in K : (x, y) \in A \} \) is convex in \( K \).

Then there exists an \( x_0 \in K \) such that \( K \times \{ x_0 \} \subset A \).

Throughout this paper, we denote by \( \langle y, x \rangle \) the duality mapping between elements \( y \in X^* \) and \( x \in X \).

2. In case of single-valued mappings

Now we consider variational-type inequalities for single-valued mappings.

Theorem 2. 1. Let \( K \) be a nonempty closed convex and bounded subset of a reflexive Banach space \( X \) and \( X^* \) be the dual of \( X \). Assume that \( T : K \rightarrow X^* \), \( \theta : K \times K \rightarrow X \) and \( \eta : K \times K \rightarrow \mathbb{R} \) are mappings satisfying the following conditions:

(1) \( \langle T(x), \theta(x, x) \rangle + \eta(x, x) = 0 \) for each \( x \in K \),
(2) the mapping
\[
x \mapsto \langle T(y), \theta(x, y) \rangle + \eta(y, x)
\]
of \( K \) into \( \mathbb{R} \) is convex for each \( y \in K \),
(3) the mapping
\[
y \mapsto \langle T(y), \theta(x, y) \rangle + \eta(y, x)
\]
of \( K \) into \( \mathbb{R} \) is continuous for each \( x \in K \).

Then there exists an \( x_0 \in K \) such that for all \( y \in K \)
\[
\langle T(x_0), \theta(y, x_0) \rangle + \eta(x_0, y) \geq 0.
\]

Proof. Let \( A := \{(x, y) \in K \times K : \langle T(y), \theta(x, y) \rangle + \eta(y, x) \geq 0 \} \), then it is easily shown that \( (x, x) \in A \). For each fixed \( x \in K \),
\[
A_x := \{ y \in K : (x, y) \in A \}
\]
\[
= \{ y \in K : \langle T(y), \theta(x, y) \rangle + \eta(y, x) \geq 0 \}
\]
is closed. Indeed, let \( \{ y_\lambda \} \) be a net in \( A_x \) such that \( y_\lambda \rightarrow y_0 \). Since \( y_\lambda \in A_x \),
we have
\[ \langle T(y_x), \theta(x, y_x) \rangle + \eta(y_x, x) \geq 0. \]

Hence by the condition (3),
\[ \langle T(y_x), \theta(x, y_x) \rangle + \eta(y_x, x) \to \langle T(y_0), \theta(x, y_0) \rangle + \eta(y_0, x). \]

Thus
\[ \langle T(y_0), \theta(x, y_0) \rangle + \eta(y_0, x) \geq 0. \]

Hence \( y_0 \in A_x \) and \( A_x \) is closed.

On the other hand, for each fixed \( y \in K \),
\[ A^y := \{x \in K: (x, y) \in A\} \]
\[ = \{x \in K: \langle T(y), \theta(x, y) \rangle + \eta(y, x) < 0\} \]
is convex. In fact, let \( x_1, x_2 \in A^y \), \( a \in [0, 1] \) and \( z = ax_1 + (1-a)x_2 \), then by the condition (2),
\[ \langle T(y), \theta(z, y) \rangle + \eta(y, z) \]
\[ = \langle T(y), \theta(ax_1 + (1-a)x_2, y) \rangle + \eta(y, ax_1 + (1-a)x_2) \]
\[ \leq a [ \langle T(y), \theta(x_1, y) \rangle + \eta(y, x_1) ] + (1-a) [ \langle T(y), \theta(x_2, y) \rangle + \eta(y, x_2) ] \]
\[ < 0, \]

hence \( z \in A^y \) and \( A^y \) is convex. Thus by Theorem 1.1, there exists an \( x_0 \in K \) such that \( K \times \{x_0\} \subset A \). This implies that there exists an \( x_0 \in K \) such that
\[ \langle T(x_0), \theta(y, x_0) \rangle + \eta(x_0, y) \geq 0, \]

for all \( y \in K \).

Remark 2.2. We obtained the same result under different conditions in [3].

3. In case of set-valued mappings

Definition 3.1[2]. Let \( X, Y \) be two topological vector spaces and \( T: X \to 2^Y \) be a set-valued mapping. \( T \) is said to be upper semicontinuous (briefly, u.s.c.) at \( x_0 \in X \) if for any open neighbourhood \( N \) containing \( T(x_0) \) there exists a neighbourhood \( M \) of \( x_0 \) such that \( T(M) \subset N \). \( T \) is said to be u.s.c. if \( T \) is u.s.c. at every point \( x \in X \).
Definition 3.2[6]. Let $X, Y$ be two topological vector spaces and $T : X \to 2^Y$ be a set-valued mapping. $T$ is said to be closed at $x \in X$ if for each nets $(x_\lambda)$ converging to $x$ and $(y_\lambda)$ converging to $y$ such that $y_\lambda \in T(x_\lambda)$ for all $\lambda$, we have $y \in T(x)$. $T$ is said to be closed if it is closed at every point $x \in X$.

Lemma 3.1[1]. Let $X, Y$ be two topological vector spaces and $T : X \to 2^Y$ be a set-valued mapping.

1) if $K$ is a compact subset of $X$, and $T$ is u.s.c. and compact-valued, then $T(K)$ is compact.

2) if $T$ is u.s.c. and compact-valued, then $T$ is closed.

Now we consider variational-type inequalities for set-valued mappings.

Theorem 3.2. Let $K$ be a nonempty closed convex and bounded subset of a reflexive Banach space $X$ and $X^*$ be the dual of $X$. Assume that $T : K \to 2^{X^*}$ is an u.s.c. mapping with compact-values, $\theta : K \times K \to X$ is a bounded mapping and $\eta : K \times K \to \mathbb{R}$ is a mapping satisfying the following conditions:

1) for each $x \in K$, there exists $t \in T(x)$ such that $\langle t, \theta(x, x) \rangle + \eta(x, x) = 0$,

2) a mapping

$$x \mapsto \langle t, \theta(x, y) \rangle + \eta(y, x)$$

of $K$ into $\mathbb{R}$ is convex for all $y \in K$ and for all $t \in T(y)$,

3) for each $x \in K$, mappings $y \mapsto \theta(x, y)$ and $y \mapsto \eta(y, x)$ are continuous.

Then there exists an $x_0 \in K$ and $t_0 \in T(x_0)$ such that for any $y \in K$

$$\langle t_0, \theta(y, x_0) \rangle + \eta(x_0, y) \geq 0.$$

Proof. Let $A = \{(x, y) \in K \times K :$ there exists $t \in T(y)$ such that $\langle t, \theta(x, y) \rangle + \eta(y, x) \geq 0 \}$, then it is easily shown that $(x, x) \in A$. For each fixed $x \in K$,

$$A_x := \{y \in K : (x, y) \in A\}$$

$$= \{y \in K :$ there exists $t \in T(y)$ such that $\langle t, \theta(x, y) \rangle + \eta(y, x) \geq 0 \}$$

is closed. Indeed, let $(y_\lambda)$ be a net in $A_x$ such that $y_\lambda \to y_0$. Since $y_\lambda \in A_x$, there exists $t_\lambda \in T(y_\lambda)$ such that $\langle t_\lambda, \theta(x, y_\lambda) \rangle + \eta(y_\lambda, x) \geq 0$. 

Since $K$ is weakly compact, by Lemma 3. 1(1), $T(K)$ is compact and hence without loss of generality, we can assume that there exists $t_0 \in T(K)$ such that $t_\lambda \to t_0$. By Lemma 3. 1(2), $T$ is closed, hence $t_0 \in T(y_0)$. By the condition (3), we have

$$\| \langle t_\lambda, \theta(x, y_\lambda) \rangle + \eta(y_\lambda, x) - (\langle t_0, \theta(x, y_0) \rangle + \eta(y_0, x) \rangle \|$$

$$\leq \| \langle t_\lambda, \theta(x, y_\lambda) \rangle - \langle t_0, \theta(x, y_0) \rangle \| + \| \eta(y_\lambda, x) - \eta(y_0, x) \|$$

$$\leq \| t_\lambda - t_0 \| \| \theta(x, y_\lambda) \| + \| t_0 \| \| \theta(x, y_\lambda) - \theta(x, y_0) \| + \| \eta(y_\lambda, x) - \eta(y_0, x) \|$$

$$\to 0 \text{ as } \lambda \to \infty.$$ Consequently, there exists $t_0 \in T(y_0)$ such that $\langle t_0, \theta(x, y_0) \rangle + \eta(y_0, x) \geq 0$.

Hence $y_0 \in A_x$ and $A_x$ is closed.

On the other hand, for each fixed $y \in K$,

$$A^y := \{ x \in K: (x, y) \in A \}$$

$$= \{ x \in K: \text{ for all } t \in T(y), \langle t, \theta(x, y) \rangle + \eta(y, x) < 0 \}$$

is convex. In fact, let $x_1, x_2 \in A^y$, $a \in [0, 1]$ and $z = ax_1 + (1-a)x_2$, then for all $t \in T(y)$,

$$\langle t, \theta(z, y) \rangle + \eta(y, z)$$

$$= \langle t, \theta(ax_1 + (1-a)x_2, y) \rangle + \eta(y, ax_1 + (1-a)x_2)$$

$$\leq a \left[ \langle t, \theta(x_1, y) \rangle + \eta(y, x_1) \right] + (1-a) \left[ \langle t, \theta(x_2, y) \rangle + \eta(y, x_2) \right]$$

$$< 0,$$

hence $z \in A^y$. By Theorem 1. 1, there exists an $x_0 \in K$ such that $K \times \{ x_0 \} \subseteq A$. This implies that there exists an $x_0 \in K$ and $t_0 \in T(x_0)$ such that for all $y \in K$, $\langle t_0, \theta(y, x_0) \rangle + \eta(x_0, y) \geq 0$.

**Corollary 3. 3.** Considering $T: K \to X^*$ in Theorem 3. 2, we obtain Theorem 2. 1 as a corollary.

**References**


