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MORITA EQUIVALENCE FOR NONCOMMUTATIVE TORI

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ABSTRACT. We give an easy proof of the fact that every noncommutative torus A_ω is stably isomorphic to the noncommutative torus $C(\widehat{S_\omega}) \otimes A_\rho$ which has a trivial bundle structure. It is well known that stable isomorphism of two separable C^*-algebras is equivalent to the existence of equivalence bimodule between them, and we construct a concrete equivalence bimodule between the two stably isomorphic C^*-algebras A_ω and $C(\widehat{S_\omega}) \otimes A_\rho$.

1. Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$. Especially, the twisted group C^*-algebra $C^*(\mathbb{Z}^l, \omega)$ is said to be a *noncommutative torus of rank* l and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier ω is called *totally skew* if the symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew (see [1, 4, 5]). It was shown in [1] that if G is a locally compact abelian group and ω is a totally skew multiplier on G then the restriction of ω-representations of G to S_ω induces a canonical homeomorphism of $\mathrm{Prim}(C^*(G, \omega))$ with $\widehat{S_\omega}$, and thus if ω is totally skew on G then $C^*(G, \omega)$ is a simple C^*-algebra.

The noncommutative torus A_ω of rank l is obtained by an iteration of $l-1$ crossed products by actions of $\mathbb{Z}$, the first action on

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$C(\mathbb{T}^1)$. When A_ω is not simple, by a change of basis, A_ω can be obtained by an iteration of $l-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{\frac{m}{k}}$, where the actions on the fibre $M_k(\mathbb{C})$ of $A_{\frac{m}{k}}$ are trivial, since $M_k(\mathbb{C})$ is a factor of the fibre of A_ω. So one can assume that A_ω is given by twisting $C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2})$ in $A_{\frac{m}{k}} \otimes C^*(\mathbb{Z}^{l-2})$ by the restriction of the multiplier ω to $k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}$, where $\widehat{k\mathbb{Z} \times k\mathbb{Z}}$ is the primitive ideal space of $A_{\frac{m}{k}}$ and $C^*(k\mathbb{Z} \times k\mathbb{Z}, \text{res of } \omega) = C^*(k\mathbb{Z} \times k\mathbb{Z})$. It is well known (cf. [4,5]) that A_ω is realized as a C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\text{Prim}(A_\omega) = \widehat{S_\omega}$ with fibres $C^*(\mathbb{Z}^l/S_\omega, \omega_1)$ for ω_1 a suitable totally skew multiplier on $\mathbb{Z}^l/S_\omega$. Poguntke proved in [5] that A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^l/S_\omega, \omega_1)$. In [3], the authors showed that two separable C^*-algebras A and B are stably isomorphic if and only if they are strongly Morita equivalent, i.e., there exists an A-B-equivalence bimodule defined in the next section. Thus A_ω is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^l/S_\omega, \omega_1)$. And Brabanter ([2]) constructed an $A_{\frac{m}{k}}$-$C(\mathbb{T}^2)$-equivalence bimodule. Modifying his construction, we are going to construct an A_ω-$C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^l/S_\omega, \omega_1)$-equivalence bimodule.

2. Morita equivalence for noncommutative tori

Rieffel introduced the concept of strong Morita equivalence for C^*-algebras.

DEFINITION 1 ([6]). Let A and B be C^*-algebras. By an A-B-*equivalence bimodule* is meant an A-B-bimodule X on which are defined an A-valued and a B-valued inner product such that

i) $\langle x, y\rangle_A z = x\langle y, z\rangle_B, \quad \forall x, y, z \in X$,
ii) the representation of A on X is a continuous $*$-representation by operators which are bounded for $\langle\cdot,\cdot\rangle_B$, and similarly for the right representation of B,
iii) the linear span of $\langle X, X\rangle_B$, which is an ideal in B, is dense in B, and similarly for $\langle X, X\rangle_A$.

We say that two C^*-algebras A and B are *strongly Morita equivalent* if there exists an A-B-equivalence bimodule.

LEMMA 2 ([2, Proposition 1]). *The rational rotation algebra* $A_{\frac{m}{k}}$ *is isomorphic to the* C^**-algebra of matrices* $(f_{ij})_{i,j=1}^{k}$ *of functions* f_{ij} *with*

$$\begin{aligned} &f_{ij} \in C^*(k\mathbb{Z} \times k\mathbb{Z}) \text{ if } i,j \in \{1,2,\cdots,k-1\} \text{ or } (i,j)=(k,k) \\ &f_{ik} \in \Omega \text{ if } i \in \{1,2,\cdots,k-1\} \\ &f_{ki} \in \Omega^* \text{ if } i \in \{1,2,\cdots,k-1\}, \end{aligned}$$

where Ω *and* Ω^* *are the* $C^*(k\mathbb{Z} \times k\mathbb{Z})$*-modules defined as*

$$\begin{aligned} \Omega &= \{f \in C(\widehat{k\mathbb{Z}} \times [0,1]) \mid f(z,1) = z^s f(z,0), \quad \forall z \in \widehat{k\mathbb{Z}}\} \\ \Omega^* &= \{f \in C(\widehat{k\mathbb{Z}} \times [0,1]) \mid f^* \in \Omega\} \end{aligned}$$

for an integer s *such that* $sm = 1$ *(mod* k*).*

When A_ω is not simple, by a change of basis, the noncommutative torus A_ω of rank l is obtained by an iteration of $l-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{\frac{m}{k}}$. Since the fibre $M_k(\mathbb{C})$ of $A_{\frac{m}{k}}$ is a factor of the fibre of A_ω, A_ω can be obtained by an iteration of $l-2$ crossed products by actions of $\mathbb{Z}$, the first action on $A_{\frac{m}{k}}$, where the actions on the fibre $M_k(\mathbb{C})$ are trivial. So one can assume that A_ω is given by twisting $C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2})$ in $A_{\frac{m}{k}} \otimes C^*(\mathbb{Z}^{l-2})$ by the restriction of the multiplier ω to $k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}$, where $\widehat{k\mathbb{Z} \times k\mathbb{Z}}$ is the primitive ideal space of $A_{\frac{m}{k}}$ and $C^*(k\mathbb{Z} \times k\mathbb{Z}, \text{res of } \omega) = C^*(k\mathbb{Z} \times k\mathbb{Z})$. Let A_ω be a noncommutative torus with fibres $C^*(\mathbb{Z}^l/S_\omega, \omega_1)$ for a suitable totally skew multiplier ω_1 on $\mathbb{Z}^l/S_\omega$. Recently, Poguntke ([5]) proved that A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^l/S_\omega, \omega_1)$. The Mackey machine for a twisted crossed product says that $C^*(\mathbb{Z}^l/S_\omega, \omega_1)$ is isomorphic to the tensor product of a completely irrational noncommutative torus A_ρ with a matrix algebra $M_{kd}(\mathbb{C})$. We are going to give an easy proof.

THEOREM 3 ([5]). *Let* A_ω *be a noncommutative torus given as above. Then* A_ω *is stably isomorphic to* $C(\widehat{S_\omega}) \otimes A_\rho \otimes M_{kd}(\mathbb{C})$ *for* A_ρ *a completely irrational noncommutative torus.*

Proof. By the Brabanter theorem ([2, Theorem 3]), $A_{\frac{m}{k}}$ is strongly Morita equivalent to $C^*(k\mathbb{Z} \times k\mathbb{Z})$. So by [3, Theorem 1.2], $A_{\frac{m}{k}}$ is stably isomorphic to $C^*(k\mathbb{Z} \times k\mathbb{Z}) \otimes M_k(\mathbb{C})$. A_ω is realized as the crossed product

$$A_{\frac{m}{k}} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_l} \mathbb{Z},$$

where α_i act trivially on the fibre $M_k(\mathbb{C})$ of $A_{\frac{m}{k}}$. But

$$\begin{aligned} A_\omega \otimes \mathcal{K}(\mathcal{H}) &\cong (A_{\frac{m}{k}} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_l} \mathbb{Z}) \otimes \mathcal{K}(\mathcal{H}) \\ &\cong (A_{\frac{m}{k}} \otimes \mathcal{K}(\mathcal{H})) \times_{\widetilde{\alpha_3}} \mathbb{Z} \times_{\widetilde{\alpha_4}} \cdots \times_{\widetilde{\alpha_l}} \mathbb{Z}, \end{aligned}$$

where $\widetilde{\alpha_i}$ are the canonical extensions of α_i such that $\widetilde{\alpha_i}$ act trivially on $M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$. Thus

$$\begin{aligned} A_\omega \otimes \mathcal{K}(\mathcal{H}) &\cong (C(\mathbb{T}^2) \otimes M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})) \times_{\widetilde{\alpha_3}} \mathbb{Z} \times_{\widetilde{\alpha_4}} \cdots \times_{\widetilde{\alpha_l}} \mathbb{Z} \\ &\cong (C(\mathbb{T}^2) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_l} \mathbb{Z}) \otimes M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H}). \end{aligned}$$

So A_ω is stably isomorphic to $(C(\mathbb{T}^2) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_l} \mathbb{Z}) \otimes M_k(\mathbb{C}) \cong C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}, \text{res of } \omega) \otimes M_k(\mathbb{C})$. Now $C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}, \text{res of } \omega)$ is a noncommutative torus with fibres $A_\rho \otimes M_d(\mathbb{C})$. So by a finite step of the above process, one can obtain the result. Therefore, A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes A_\rho \otimes M_{kd}(\mathbb{C})$. □

One can construct an equivalence bimodule to obtain the result.

THEOREM 4. *A_ω is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes A_\rho$.*

Proof. A_ω can be given by twisting $C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2})$ in $A_{\frac{m}{k}} \otimes C^*(\mathbb{Z}^{l-2})$ by the restriction of the multiplier ω to $k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}$. So A_ω is given by canonically replacing $C^*(k\mathbb{Z} \times k\mathbb{Z})$ in the (right) $C^*(k\mathbb{Z} \times k\mathbb{Z})$-modules in the matrix representation of $A_{\frac{m}{k}}$ given in Lemma 2 by $C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}, \text{res of } \omega)$. Let $A_{r(\omega)}$ be the noncommutative torus $C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}, \text{res of } \omega)$. Then A_ω is isomorphic to the C^*-algebra of matrices $(g_{ij})_{i,j=1}^k$ of g_{ij} with

$$\begin{aligned} &g_{ij} \in A_{r(\omega)} \text{ if } i, j \in \{1, 2, \cdots, k-1\} \text{ or } (i,j) = (k,k) \\ &g_{ik} \in \Gamma \text{ if } i \in \{1, 2, \cdots, k-1\} \\ &g_{kj} \in \Gamma^* \text{ if } j \in \{1, 2, \cdots, k-1\}, \end{aligned}$$

where Γ and Γ^* are the $C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}, \text{res of } \omega)$-modules given by canonically replacing $C^*(k\mathbb{Z} \times k\mathbb{Z})$ in the $C^*(k\mathbb{Z} \times k\mathbb{Z})$-modules Ω and Ω^* given in the statement of Lemma 2 by $C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}, \text{res of } \omega)$.

Let X be the complex vector space $(\oplus_1^{k-1}\Gamma) \oplus A_{r(\omega)}$. We will consider the elements of X as $(k, 1)$ matrices where the first $(k-1)$ entries are in Γ and the last entry is in $A_{r(\omega)}$. If $x \in X$, denote by x^* the $(1, k)$ matrix resulting from x by transposition and involution so that $x^* \in (\oplus_1^{k-1}\Gamma^*) \oplus A_{r(\omega)}$. The space X is a left A_ω-module if module multiplication is defined by matrix multiplication $F \cdot x$, where $F = (g_{ij})_{i,j=1}^k \in A_\omega$ and $x \in X$. If $g \in A_{r(\omega)}$ and $x \in X$, then $x \cdot [g]$ defines a right $A_{r(\omega)}$-module structure on X. Now we define an A_ω-valued inner product $\langle \cdot, \cdot \rangle_{A_\omega}$ on X and an $A_{r(\omega)}$-valued inner product $\langle \cdot, \cdot \rangle_{A_{r(\omega)}}$ on X by

$$\langle x, y \rangle_{A_\omega} = x \cdot y^* \qquad \& \qquad \langle x, y \rangle_{A_{r(\omega)}} = x^* \cdot y$$

if $x, y \in X$ and we have matrix multiplication on the right. By the same reasoning as the proof given by [2, Theorem 3], equipped with this structure, X becomes an A_ω-$A_{r(\omega)}$-equivalence bimodule. So A_ω is strongly Morita equivalent to $A_{r(\omega)} \cong C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}, \text{res of } \omega)$. Now $C^*(k\mathbb{Z} \times k\mathbb{Z} \times \mathbb{Z}^{l-2}, \text{res of } \omega)$ is a noncommutative torus with fibres $A_\rho \otimes M_d(\mathbb{C})$. So by a finite step of the above process, one can obtain the result.

Therefore, A_ω is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes A_\rho$. □

We have obtained that the noncommutative torus A_ω is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes A_\rho$, which is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes A_\rho \otimes M_{kd}(\mathbb{C}) \cong C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^l/S_\omega, \omega_1)$. So A_ω is strongly Morita equivalent to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^l/S_\omega, \omega_1)$.

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