Bull. Korean Math. Soc. **37** (2000), No. 2, pp. 255–263

COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM VARIABLES (II)

Soo Hak Sung

Abstract. Let $\{X_{nk},\ u_n \le k \le v_n,\ n \ge 1\}$ be an array of rowwise independent, but not necessarily identically distributed, random variables with $EX_{nk} = 0$ for all k and n. In this paper, we povide a domination condition under which $\sum_{k=u_n}^{v_n} X_{nk}/n^{1/p}$, $1 \le p < 2$, converges completely to zero.

1. Introduction

Hsu and Robbins (1947) introduced the concept of complete convergence. A sequence $\{X_n,\ n \ge 1\}$ of random variables is said to converge completely to the constant C if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \epsilon) < \infty,\ \forall \epsilon > 0.$$

They also proved that if $\{X_n\}$ is a sequence of independent and identically distributed random variables with $EX_1 = 0$ and $EX_1^2 < \infty$, then S_n/n converges completely to zero, where $S_n = X_1 + \cdots + X_n$.

Now let $\{X_{nk},\ 1 \le k \le n,\ n \ge 1\}$ be an array of rowwise independent random variables such that $EX_{nk} = 0$ for all k and n. When the array $\{X_{nk}\}$ is independent and identically distributed random variables, it is easily shown that if $E|X_{11}|^{2p} < \infty$ for some $1 \le p < 2$, then $\sum_{k=1}^{n} X_{nk}/n^{1/p}$ converges completely to zero. This result has been generalized and extended in several directions. Throughout this paper, we

Received June 2, 1999.

2000 Mathematics Subject Classification: 60F15.

Key words and phrases: complete convergence, arrays, rowwise independent random variables, moving average.

The author wishes to acknowledge the financial support of the Korea Research Foundation made in the program year of 1998 (1998-001-D00144).

only consider the extension to arrays of rowwise independent, but not necessarily identically distributed, random variables.
An array $\{X_{nk}\}$ is said to be uniformly bounded by a random variable X if

(1) $P(|X_{nk}| > t) \leq P(|X| > t)$ for all $t \geq 0$ and for all k and n.

Hu, Móricz, and Taylor (1989) proved that if $\{X_{nk}\}$ is an array of rowwise independent random variables satisfying $EX_{nk} = 0$ and (1) with X such that $E|X|^{2p} < \infty$ for some $1 \leq p < 2$, then $S_n/n^{1/p}$ converges completely to zero, where $S_n = \sum_{k=1}^{n} X_{nk}$.
Gut (1992) introduced the concept of weakly mean domination. An array $\{X_{nk}\}$ is said to be weakly mean dominated by a random variable X if for some $C > 0$

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_{nk}| > t) \leq CP(|X| > t) \text{ for all } t \geq 0 \text{ and all } n. \tag{2}$$

Note that the condition (1) implies the condition (2). He also proved Hu, Móricz, and Taylor's (1989) theorem under the weaker condition (2).

For the more general array $\{X_{nk},\ u_n \leq k \leq v_n,\ n \geq 1\}$, we introduce the following domination condition. For $p > 0$,

$$\frac{1}{n}\sum_{k=u_n}^{v_n} E|X_{nk}|^p I(|X_{nk}|^p > t) \leq CE|X|^p I(|X|^p > t) \quad \text{for all } t \geq 0 \text{ and all } n. \tag{3}$$

When $u_n = 1,\ v_n = n$ for $n \geq 1$, the condition (3) is weaker than the condition (2) (see the proof of Corollary 1).

In this paper, we extend Gut's (1992) theorem to the array $\{X_{nk},\ u_n \leq k \leq v_n\}$ satisfying (3). From this result, we obtain a complete convergence result for moving average processes.

Throughout this paper, C denotes a positive constant which may be different in various places.

2. Main result

To prove the main result, we will need the following lemmas.

LEMMA 1. *For* $r > 0$

$$\sum_{n=1}^{\infty} E|X|^r I(|X|^r > n) \le E|X|^{2r}.$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} E|X|^r I(|X|^r > n) &= \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} E|X|^r I(k < |X|^r \le k+1)\\ &= \sum_{k=1}^{\infty} E|X|^r I(k < |X|^r \le k+1)k\\ &\le E|X|^{2r}.\end{aligned}$$

□

The following lemma is well known. In the first inequality, one has $C \le 2$ (see von Bahr and Esseen (1965)). For the second inequality, see Rosenthal (1970).

LEMMA 2. *Let* $X_1.\cdots, X_n$ *be independent random variables with* $EX_k = 0$ *for* $1 \le k \le n$. *Then the following statememts hold.*
(i) $E|\sum_{k=1}^{n} X_k|^r \le C\sum_{k=1}^{n} E|X_k|^r$ *if* $1 \le r \le 2$.
(ii) $E|\sum_{k=1}^{n} X_k|^r \le C\{\sum_{k=1}^{n} E|X_k|^r + (\sum_{k=1}^{n} E|X_k|^2)^{r/2}\}$ *if* $r > 2$.

LEMMA 3. *Let* $0 < p < \alpha$ *and suppose that* $\{X_{nk}, u_n \le k \le v_n\}$ *is an array of rowwise independent random variables satisfying* (3). *Then*

$$\begin{aligned}&\sum_{k=u_n}^{v_n} E|X_{nk}|^{\alpha} I(|X_{nk}|^p \le n)\\ &\le C\left\{nE|X|^p + n\sum_{i=1}^{n} i^{\frac{\alpha}{p}-2} E|X|^p I(|X|^p > i)\right\}.\end{aligned}$$

Proof. By using the mean value theorem, we obtain

$$(4) \qquad (i+1)^{\frac{\alpha}{p}-1} - i^{\frac{\alpha}{p}-1} \le \begin{cases} (\frac{\alpha}{p}-1)(i+1)^{\frac{\alpha}{p}-2} & \text{if } \frac{\alpha}{p} > 2 \\ (\frac{\alpha}{p}-1)i^{\frac{\alpha}{p}-2} & \text{if } 1 < \frac{\alpha}{p} \le 2. \end{cases}$$

Since $0 < p < \alpha$, it follows by (3) and (4) that

$$\begin{aligned}
&\sum_{k=u_n}^{v_n} E|X_{nk}|^\alpha I(|X_{nk}|^p \le n) \\
&= \sum_{k=u_n}^{v_n} \sum_{i=1}^{n} E|X_{nk}|^\alpha I(i-1 < |X_{nk}|^p \le i) \\
&\le \sum_{k=u_n}^{v_n} \sum_{i=1}^{n} i^{\frac{\alpha}{p}-1} E|X_{nk}|^p I(i-1 < |X_{nk}|^p \le i) \\
&= \sum_{k=u_n}^{v_n} \sum_{i=1}^{n} i^{\frac{\alpha}{p}-1} \Big[E|X_{nk}|^p I(|X_{nk}|^p > i-1) - E|X_{nk}|^p I(|X_{nk}|^p > i) \Big] \\
&= \sum_{k=u_n}^{v_n} \Big[E|X_{nk}|^p I(|X_{nk}|^p > 0) - n^{\frac{\alpha}{p}-1} E|X_{nk}|^p I(|X_{nk}|^p > n) \\
&\quad + \sum_{i=1}^{n-1} ((i+1)^{\frac{\alpha}{p}-1} - i^{\frac{\alpha}{p}-1}) E|X_{nk}|^p I(|X_{nk}|^p > i) \Big] \\
&\le \sum_{k=u_n}^{v_n} E|X_{nk}|^p I(|X_{nk}|^p > 0) \\
&\quad + \sum_{i=1}^{n-1} ((i+1)^{\frac{\alpha}{p}-1} - i^{\frac{\alpha}{p}-1}) \sum_{k=u_n}^{v_n} E|X_{nk}|^p I(|X_{nk}|^p > i) \\
&\le C \Big\{ nE|X|^p I(|X|^p > 0) + n \sum_{i=1}^{n-1} ((i+1)^{\frac{\alpha}{p}-1} - i^{\frac{\alpha}{p}-1}) E|X|^p I(|X|^p > i) \Big\} \\
&\le C \Big\{ nE|X|^p + n \sum_{i=1}^{n} i^{\frac{\alpha}{p}-2} E|X|^p I(|X|^p > i) \Big\}.
\end{aligned}$$

□

Now we state and prove our main result.

THEOREM 1. *Let $1 \le p < 2$ and suppose that $\{X_{nk},\ u_n \le k \le v_n,\ n \ge 1\}$ is an array of rowwise independent random variables satisfying $EX_{nk} = 0$ and (3) with X such that $E|X|^{2p} < \infty$. Then $\sum_{k=u_n}^{v_n} X_{nk}/n^{1/p}$ converges completely to zero.*

Proof. Let $X'_{nk} = X_{nk}I(|X_{nk}|^p \le n)$ and $X''_{nk} = X_{nk}I(|X_{nk}|^p > n)$. Since $EX_{nk} = 0$, it follows that

$$\frac{\sum_{k=u_n}^{v_n} X_{nk}}{n^{1/p}} = \frac{\sum_{k=u_n}^{v_n}(X'_{nk} - EX'_{nk})}{n^{1/p}} + \frac{\sum_{k=u_n}^{v_n}(X''_{nk} - EX''_{nk})}{n^{1/p}}.$$

Hence, it is enough to show that

$$\frac{\sum_{k=u_n}^{v_n}(X'_{nk} - EX'_{nk})}{n^{1/p}} \to 0 \text{ completely} \tag{5}$$

and

$$\frac{\sum_{k=u_n}^{v_n}(X''_{nk} - EX''_{nk})}{n^{1/p}} \to 0 \text{ completely.} \tag{6}$$

By Lemma 2, c_r-inequality, and the condition (3), we have

$$\begin{aligned} E\left|\frac{\sum_{k=u_n}^{v_n}(X''_{nk} - EX''_{nk})}{n^{1/p}}\right|^p &\le C\frac{1}{n}\sum_{k=u_n}^{v_n} E|X''_{nk} - EX''_{nk}|^p \\ &\le C\frac{1}{n}\sum_{k=u_n}^{v_n} E|X''_{nk}|^p \\ &\le CE|X|^p I(|X|^p > n). \end{aligned}$$

It follows by Lemma 1 that

$$\begin{aligned} \sum_{n=1}^{\infty} E\left|\frac{\sum_{k=u_n}^{v_n}(X''_{nk} - EX''_{nk})}{n^{1/p}}\right|^p &\le C\sum_{n=1}^{\infty} E|X|^p I(|X|^p > n) \\ &\le CE|X|^{2p} < \infty, \end{aligned}$$

which implies that (6) holds.

Now we show that (5) holds. Let $\alpha > \frac{2p}{2-p}$. Then $\alpha > 2p$ since $1 \le p < 2$. From Lemma 2 and c_r-inequality, we have

$$E\left|\frac{\sum_{k=u_n}^{v_n}(X'_{nk} - EX'_{nk})}{n^{1/p}}\right|^\alpha$$

$$\le C\frac{1}{n^{\alpha/p}}\left\{\sum_{k=u_n}^{v_n} E|X'_{nk}|^\alpha + \left(\sum_{k=u_n}^{v_n} E|X'_{nk}|^2\right)^{\alpha/2}\right\}. \tag{7}$$

By using Lemma 1 and Lemma 3, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^{\alpha/p}}\sum_{k=u_n}^{v_n} E|X'_{nk}|^\alpha \\
&\le C\left\{E|X|^p\sum_{n=1}^{\infty}\frac{n}{n^{\alpha/p}} + \sum_{n=1}^{\infty}\frac{n}{n^{\alpha/p}}\sum_{i=1}^{n} i^{\frac{\alpha}{p}-2}E|X|^p I(|X|^p > i)\right\} \\
&= C\left\{E|X|^p\sum_{n=1}^{\infty}\frac{n}{n^{\alpha/p}} + \sum_{i=1}^{\infty}E|X|^p I(|X|^p > i) i^{\frac{\alpha}{p}-2}\sum_{n=i}^{\infty}\frac{n}{n^{\alpha/p}}\right\} \\
&\le C\{E|X|^p + \sum_{i=1}^{\infty}E|X|^p I(|X|^p > i)\} \\
&\le C\{E|X|^p + E|X|^{2p}\} < \infty.
\end{aligned} \tag{8}$$

Also, we have by Lemma 1 and Lemma 3 that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^{\alpha/p}}\left(\sum_{k=u_n}^{v_n} E|X'_{nk}|^2\right)^{\alpha/2} \\
&\le C\sum_{n=1}^{\infty}\frac{1}{n^{\alpha/p}}\left(nE|X|^p + n\sum_{i=1}^{n} i^{\frac{2}{p}-2}E|X|^p I(|X|^p > i)\right)^{\alpha/2} \\
&\le C\sum_{n=1}^{\infty}\frac{1}{n^{\alpha/p}}\left(nE|X|^p + n\sum_{i=1}^{n} E|X|^p I(|X|^p > i)\right)^{\alpha/2}
\end{aligned} \tag{9}$$

$$\leq C \sum_{n=1}^{\infty} \frac{1}{n^{\alpha/p}} (nE|X|^p + nE|X|^{2p})^{\alpha/2}$$
$$= C(E|X|^p + E|X|^{2p})^{\alpha/2} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha(\frac{1}{p}-\frac{1}{2})}} < \infty,$$

since $\alpha(\frac{1}{p} - \frac{1}{2}) > 1$. Thus, (5) holds by (7), (8), and (9). □

The following corollary was proved by Gut (1992).

COROLLARY 1. *Let $1 \leq p < 2$ and suppose that $\{X_{nk},\ 1 \leq k \leq n,\ n \geq 1\}$ is an array of rowwise independent random variables satisfying $EX_{nk} = 0$ and (2) with X such that $E|X|^{2p} < \infty$. Then $\sum_{k=1}^{n} X_{nk}/n^{1/p}$ converges completely to zero.*

Proof. By Theorem 1, it is enough to show that the condition (2) implies the condition (3) when $u_n = 1, v_n = n$. Observe that

$$E|X|^p I(|X|^p > t) = tP(|X|^p > t) + \int_t^{\infty} P(|X|^p > x)\, dx.$$

Hence we have

$$\begin{aligned}
&\frac{1}{n} \sum_{k=1}^{n} E|X_{nk}|^p I(|X_{nk}|^p > t) \\
&= \frac{1}{n} \sum_{k=1}^{n} \Big[tP(|X_{nk}|^p > t) + \int_t^{\infty} P(|X_{nk}|^p > x)\, dx \Big] \\
&= t\frac{1}{n} \sum_{k=1}^{n} P(|X_{nk}|^p > t) + \int_t^{\infty} \frac{1}{n} \sum_{k=1}^{n} P(|X_{nk}|^p > x)\, dx \\
&\leq C\Big[tP(|X|^p > t) + \int_t^{\infty} P(|X|^p > x)\, dx \Big] \\
&= CE|X|^p I(|X|^p > t).
\end{aligned}$$

□

COROLLARY 2. *Let $1 \le p < 2$ and suppose that $\{Y_k,\ -\infty < k < \infty\}$ is a doubly infinite sequence of independent random variables satisfying $EY_k = 0$ and (10) with Y such that $E|Y|^{2p} < \infty$.*

(10) $E|Y_k|^p I(|Y_k|^p > t) \le CE|Y|^p I(|Y|^p > t)$ *for all* $t \ge 0$ *and all* k.

Let $\{a_k, -\infty < k < \infty\}$ be an absolutely summable sequence of real numbers and

$$X_i = \sum_{k=-\infty}^{\infty} a_{i+k} Y_k,\ i \ge 1.$$

Then $\sum_{i=1}^n X_i/n^{1/p}$ converges completely to zero.

Proof. Obseve that

$$\sum_{i=1}^n X_i = \sum_{k=-\infty}^{\infty} \sum_{i=1}^n a_{i+k} Y_k.$$

Set $a_{nk} = \sum_{i=1}^n a_{i+k}$ and $X_{nk} = a_{nk} Y_k$. Then $\{X_{nk}, -\infty < k < \infty, n \ge 1\}$ is an array of rowwise independent random variables with $EX_{nk} = 0$. Since $\{a_k,\ -\infty < k < \infty\}$ is absolutely summable, say $\sum_{k=-\infty}^{\infty} |a_k| = C$, we have that $|a_{nk}| \le C$ and $\sum_{k=-\infty}^{\infty} |a_{nk}| \le \sum_{i=1}^n \sum_{k=-\infty}^{\infty} |a_{i+k}| \le Cn$. Then it follows by (10) that

$$\begin{aligned}
&\frac{1}{n} \sum_{k=-\infty}^{\infty} E|X_{nk}|^p I(|X_{nk}|^p > t) \\
&\le \frac{1}{n} \sum_{k=-\infty}^{\infty} |a_{nk}|^p E|Y_k|^p I\Big(|Y_k|^p > \frac{t}{C^p}\Big) \\
&\le CE|Y|^p I\Big(|Y|^p > \frac{t}{C^p}\Big) \frac{1}{n} \sum_{k=-\infty}^{\infty} |a_{nk}|^p \\
&\le CE|Y|^p I\Big(|Y|^p > \frac{t}{C^p}\Big) \max_k |a_{nk}|^{p-1} \frac{1}{n} \sum_{k=-\infty}^{\infty} |a_{nk}| \\
&\le CE|Y|^p I\Big(|Y|^p > \frac{t}{C^p}\Big).
\end{aligned}$$

Thus $\{X_{nk},\ -\infty < k < \infty, n \geq 1\}$ satisfies the condition (3) when $u_n = -\infty$, $v_n = \infty$, and $X = CY$, and so the corollary 2 follows from Theorem 1. □

REMARK 1. Li, Rao, Wang (1992) proved Corollary 2 under the stronger condition that $\{Y_k,\ -\infty < k < \infty\}$ is a sequence of independent and identically distributed random variables with $E|Y_1|^{2p} < \infty$. Sadeghi and Bozorgnia (1994) proved Corollary 2 under the stronger condition that $\{Y_k,\ -\infty < k < \infty\}$ is a sequence of independent random variables which is uniformly bounded by a random variable Y such that $E|Y|^{2p} < \infty$.

References

[1] B. von Bahr and C. G. Esseen, *Inequalities for the rth absolute moment of a sum of random variables,* $1 \leq r \leq 2$, Ann. Math. Statist. **36** (1965), 299-303.

[2] A. Gut, *Complete convergence for arrays*, Period. Math. Hung. **25** (1992), 51-75.

[3] P. L. Hsu and H. Robbins, *Complete convergence and the law of large numbers,* Proc. Nat. Acad. Sci. USA **33** (1947), 25-31.

[4] T. C. Hu, F. Móricz, and R. L. Taylor, *Strong laws of large numbers for arrays of rowwise independent random variables*, Acta Math. Hung. **54** (1989), 153-162.

[5] T. C. Hu, S. H. Sung, and A. I. Volodin, *Complete convergence for arrays of rowwise independent random variables*, submitted to Sankhya (1999).

[6] D. Li, M. B. Rao, and X. C. Wang, *Complete convergence of moving average processes*, Statist. Probab. Lett. **14** (1992), 111-114.

[7] H. P. Rosenthal, *On the subspace of* $L^p (p > 2)$ *spanned by sequences of independent random variables*, Israel J. Math. **8** (1970), 273-303.

[8] H. Sadeghi and A. Bozorgnia, *Moving average and complete convergence*, Bull. Iranian Math. Soc. **20** (1994), 37-42.

DEPARTMENT OF APPLIED MATHEMATICS, PAI CHAI UNIVERSITY, TAEJON 302-735, KOREA
E-mail: sungsh@www.paichai.ac.kr