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THE ASYMPTOTIC STABILITY OF SOME INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we consider two delay equations with infinite delay. We will give two sufficient conditions for the positive and zero equilibriums of these equations to be a global attractor respectively.

1. Introduction

In this note we consider the following integrodifferential equations with infinite delay

$$p'(t) + rp(t) = \alpha \int_0^\infty K(s) \frac{p^n(t-s)}{\beta + p^n(t-s)}\, ds \tag{1.1}$$

and

$$p'(t) + rp(t) = \alpha \int_0^\infty K(s) \frac{1}{1+p(t-s)}\, ds, \tag{1.2}$$

where r, α and β are positive constants; n is a positive integer and K is a nonnegative function. One of the models with bounded delay describes different periodic diseases and was proposed by M. C. Mackey and L. Glass [1]. For more details of the derivation and numerical studying of the models we can refer to the articles of M. C. Mackey and U. an der Heiden [9], L. Glass and M. C. Mackey [2], and Tsen F.-S. P. [11]. J. K. Hale and N. Sternberg [6] also gave some interesting and nice results for

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the numerical and chaotic problems of this type equation with bounded delay.

The goal of this paper is to investigate the global stability of equations (1.1) and (1.2) respectively. We obtain some sufficient conditions for the positive and zero equilibriums of equations (1.1) and (1.2) to be a global attractor respectively. The main results and the proofs will be stated in section 2.

2. Main Results

In this section we study the global asymptotic stability of the steady states of equations (1.1) and (1.2) respectively. First we consider the following integrodifferential equation.

$$p'(t) + rp(t) = \alpha \int_0^\infty K(s) \frac{p^n(t-s)}{\beta + p^n(t-s)}\, ds; \quad t > 0, \tag{2.1}$$

with the initial condition

$$p(s) = \phi(s) \geq 0; \ s \in (-\infty, 0]; \tag{2.2}$$

where r, α, β and $n \geq 1$ are positive constants. Here, we assume that ϕ is piecewise bounded continuous and K is a nonnegative kernel satisfying

$$K_1 = \int_0^\infty K(s)\, ds < \infty, \quad K_2 = \int_0^\infty sK(s)\, ds < \infty. \tag{2.3}$$

We give the following result about the globally asymptotic stability of the zero solution of Eq. (2.1).

THEOREM 2.1. *Suppose that* (2.3) *holds and* $p(t)$ *is the positive solution of* (2.1)-(2.2) *in* $[0, \infty)$. *Assume that*

$$\frac{\alpha\beta n\omega^{n-1}K_1}{(\beta + \omega^n)^2} < r, \quad \text{for} \quad n > 1, \tag{2.4}$$

$$\frac{\alpha K_1}{\beta} < r, \quad \text{for} \quad n = 1, \tag{2.5}$$

where

$$\omega = \sqrt[n]{\frac{n-1}{n+1}}\beta. \tag{2.6}$$

Then

$$\lim_{t\to\infty} p(t) = 0.$$

In order to prove Theorem 2.1, we need the following lemma.

LEMMA 2.2. *Let f be a nonnegative function defined on $[0,\infty)$ such that f is integrable on $[0,\infty)$ and uniformly continuous on $[0,\infty)$. Then* $\lim\limits_{t\to\infty} f(t) = 0$.

Proof. See, for example, Gopalsamy [3]. □

Proof of Theorem 2.1. Let $F(u) = \frac{u^n}{\beta+u^n}$. Then

$$\begin{aligned} p'(t) &= -rp(t) + \alpha\int_0^\infty K(s)[F(p(t-s)) - F(0)]\,ds \\ &= -rp(t) + \alpha\int_0^\infty K(s)Q(t,s)p(t-s)\,ds, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} Q(t,s) &= \frac{\beta n\xi^{n-1}(t,s)}{[\beta+\xi^n(t,s)]^2}, \quad n > 1 \\ &= \frac{\beta}{[\beta+\xi(t,s)]^2}, \quad n = 1, \end{aligned}$$

where $\xi(t,s)$ lies in the segment joining the two points $p(t-s)$ and 0. It is easy to see that

$$0 \le Q(t,s) \le \frac{\beta n\omega^{n-1}}{(\beta+\omega^n)^2} \quad \text{for} \quad n > 1, \quad \text{for all} \quad t \ge s, \tag{2.8}$$

$$0 \le Q(t,s) \le \frac{1}{\beta} \quad \text{for} \quad n = 1, \quad \text{for all} \quad t \ge s, \tag{2.9}$$

where ω is given in (2.6).
Define

$$V(t) = p(t) + \alpha \int_0^\infty \left[K(s) \cdot \left(\int_{t-s}^{t} Q(u+s,s)p(u)\, du \right) \right] ds. \tag{2.10}$$

From (2.1)-(2.3) and (2.8)-(2.10), it is easy to see that $V(t)$ is well-defined and $V(t) > 0$ for all $t > 0$. Then from (2.1) and (2.10), we obtain

$$\begin{aligned} V'(t) &= \left(-r + \alpha \int_0^\infty K(s)Q(t+s,s)\, ds \right) p(t) \\ &\le -c^* p(t) \\ &< 0 \quad \text{for all} \quad t \ge 0, \end{aligned} \tag{2.11}$$

where $c^* = r - \frac{\alpha\beta n\omega^{n-1}K_1}{(\beta+\omega^n)^2} > 0$ for $n > 1$; and $c^* = r - \frac{\alpha K_1}{\beta} > 0$ for $n = 1$.
Then

$$V(t) - V(0) \le -c^* \int_0^t p(s)\, ds \quad \text{for all} \quad t \ge 0$$

and thus

$$c^* \int_0^t p(s)\, ds \le V(0) < \infty \quad \text{for all} \quad t \ge 0.$$

Hence

$$0 < \int_0^\infty p(s)\, ds < \infty.$$

Since $p(t)$ is positive and bounded, by (2.1) we obtain that $|p'(t)|$ is bounded which implies that $p(t)$ is uniformly continuous on $[0, \infty)$. Thus, By Lemma 2.2,

$$\lim_{t\to\infty} p(t) = 0.$$

The proof is complete. □

Next, we consider the following delay equation

$$y'(t) + ry(t) = \alpha \int_0^\infty K(s) \cdot \frac{1}{1+y(t-s)}\, ds; \quad t > 0 \tag{2.12}$$

together with an initial condition of the form

$$y(s) = \phi(s) \geq 0;\ \ s \in (-\infty, 0]; \tag{2.13}$$

where r, α, β are positive constants, the function $K(s) \not\equiv$ *const.* is a nonnegative kernel satisfying the condition (2.3), and ϕ is a piecewise continuous function on $(-\infty, 0]$. It is clear that (2.12) has a unique constant steady state $y^* > 0$ and y^* satisfies

$$y^*(1 + y^*) = \frac{\alpha K_1}{r}. \tag{2.14}$$

Now we give a sufficient condition for the positive steady state p^* of (2.12) to be a global attractor.

THEOREM 2.3. *Assume that* (2.3) *holds,* $K_3 = \int_0^\infty s^2 K(s)\, ds < \infty$, *and*

$$r + \frac{\alpha K_1}{(1+U)^2} - \left[r\alpha + \frac{\alpha^2 K_1}{(1+L)^2} \right] \frac{K_2}{(1+L)^2} > 0, \tag{2.15}$$

where L *and* U *are defined by*

$$L = \min\left(y(0)\ ,\ \frac{\alpha K_1}{r(1+m)} \right), \qquad U = \max\left(y(0)\ ,\ \frac{\alpha K_1}{r} \right), \tag{2.16}$$

and

$$m = \max\left(\frac{\alpha K_1}{r}\ ,\ \max_{-\infty < t \leq 0} \phi(t) \right). \tag{2.17}$$

Then every positive solution of (2.12) *satisfies*

$$\lim_{t \to \infty} y(t) = y^*.$$

Proof. Let $y(t)$ be the solution of (2.12) and (2.13). We divide the proof into the following steps.

Step 1. We claim:

$$L \leq y(t) \leq U \qquad \text{for all} \qquad t \geq 0,$$

where L and U are defined in (2.16)-(2.17).

Proof of above claim. It is clear that

$$y'(t) + ry(t) \leq \alpha K_1$$

since $y(t)$ is positive and by the definition of K_1. Then

$$y(t) \leq \left(y(0) - \frac{\alpha K_1}{r}\right) e^{-rt} + \frac{\alpha K_1}{r} \leq \max\left(y(0)\ ,\ \frac{\alpha K_1}{r}\right) = U.$$

Moreover, $y'(t)+ry(t) \geq \frac{\alpha K_1}{(1+m)}$ since $y(t) \leq U \leq m$, where m is defined in (2.17). Hence

$$\begin{aligned} y(t) &\geq \left(y(0) - \frac{\alpha K_1}{r(1+m)}\right) e^{-rt} + \frac{\alpha K_1}{r(1+m)} \\ &\geq \min\left(y(0)\ ,\ \frac{\alpha K_1}{r(1+m)}\right) = L. \end{aligned}$$

The proof of above claim is complete. □

Step 2. Let $y(t) = y^* + z(t)$. Then $z(t)$ is bounded and satisfies

$$z'(t) + ry^* + rz(t) = \alpha \int_0^\infty K(s) \frac{1}{1 + y^* + z(t-s)}\, ds$$

or, by (2.14),

$$z'(t) = -rz(t) + \alpha \int_0^\infty K(s) \left[\frac{1}{1 + y^* + z(t-s)} - \frac{1}{1+y^*}\right] ds. \tag{2.18}$$

Set $F(u) = \frac{1}{1+u}$. Then (2.18) becomes

$$z'(t) = -rz(t) + \alpha \int_0^\infty K(s)\left[F(y^* + z(t-s)) - F(y^*)\right]\, ds$$

and thus

$$z'(t) = -rz(t) - \alpha \int_0^\infty K(s)Q(t,s)z(t-s)\, ds, \tag{2.19}$$

where $Q(t,s) = \frac{1}{[1+\xi(t,s)]^2}$, and $\xi(t,s)$ lies in the segment joining the two points $y^* + z(t-s)$ and y^*.

By Step 1, it is easy to see that

$$\frac{1}{(1+U)^2} \le Q(t,s) \le \frac{1}{(1+L)^2}. \tag{2.20}$$

From (2.19)-(2.20) we obtain that

$$\begin{aligned} &\frac{d}{dt}\left[z(t) - \alpha \int_0^\infty \left(K(s) \cdot \int_{t-s}^t Q(u+s,s)z(u)\, du\right) ds\right] \\ &= -\left[r + \alpha \int_0^\infty K(s)Q(t+s,s)\, ds\right] z(t) \end{aligned} \tag{2.21}$$

Now, define $V(t)$ by

(2.22)

$$V(t) = \left[z(t) - \alpha \int_0^\infty \left(K(s) \cdot \int_{t-s}^t Q(u+s,s)z(u)\, du\right) ds\right]^2 + V_1(t),$$

where V_1 is a smooth function to be determined later.

Then

$$V'(t) = -2\left[z(t) - \alpha \int_0^\infty \left(K(s)\cdot \int_{t-s}^t Q(u+s,s)z(u)\,du\right)ds\right]$$
$$\cdot\left[r + \alpha\int_0^\infty K(s)Q(t+s,s)\,ds\right]\cdot z(t) + V_1'(t)$$
$$= -2\left[r + \alpha\int_0^\infty K(s)Q(t+s,s)\,ds\right]z^2(t)$$
$$+ 2\alpha\left[r + \alpha\int_0^\infty K(s)Q(t+s,s)\,ds\right]z(t)$$
$$\cdot\int_0^\infty \left(K(s)\cdot\int_{t-s}^t Q(u+s,s)z(u)\,du\right)ds + V_1'(t).$$

By using the inequality

$$2z(t)z(u) \le z^2(t) + z^2(u)$$

we obtain that

$$V'(t) \le -2\left[r + \alpha\int_0^\infty K(s)Q(t+s,s)\,ds\right]z^2(t)$$
$$+ \alpha\left(r + \alpha\int_0^\infty K(s)Q(t+s,s)\,ds\right)z^2(t)$$
$$\cdot\int_0^\infty\left(K(s)\cdot\int_{t-s}^t Q(u+s,s)\,du\right)ds \tag{2.23}$$
$$+ \alpha\left(r + \alpha\int_0^\infty K(s)Q(t+s,s)\,ds\right)$$
$$\cdot\int_0^\infty\left(K(s)\cdot\int_{t-s}^t Q(u+s,s)z^2(u)\,du\right)ds + V_1'(t).$$

Choose

$$V_1(t) = \alpha\int_0^\infty\left\{K(s)\cdot\left[\int_{t-s}^t\left(r + \alpha\int_0^\infty K(x)Q(v+s+x,x)\,dx\right)\right.\right.$$
$$\left.\left.\cdot\left(\int_v^t Q(u+s,s)z^2(u)\,du\right)dv\right]\right\}ds. \tag{2.24}$$

Then, from Step 1, (2.20) and the conditions on K, it is easy to see that $V_1(t)$ is well-defined for all $t \geq 0$ and

$$
\begin{aligned}
V'(t) \leq & -2\left[r+\alpha\int_0^\infty K(s)Q(t+s,s)\,ds\right]z^2(t)\\
& +\alpha\left[r+\alpha\int_0^\infty K(s)Q(t+s,s)\,ds\right]z^2(t)\\
& \cdot\int_0^\infty\left(K(s)\cdot\int_{t-s}^{t} Q(u+s,s)\,du\right)ds\\
& +\alpha z^2(t)\int_0^\infty\Big\{K(s)Q(t+s,s)\\
& \cdot\int_{t-s}^{t}\left[r+\alpha\int_0^\infty K(x)Q(v+s+x,x)\,dx\right]\,dv\Big\}ds\\
\leq & -2rz^2(t)-2\alpha\int_0^\infty K(s)\frac{1}{(1+U)^2}\,ds\,z^2(t)\\
& +\alpha\left[r+\alpha\int_0^\infty K(s)\frac{.1}{(1+L)^2}\,ds\right]z^2(t)\\
& \cdot\int_0^\infty\left[K(s)\cdot\left(\int_{t-s}^{t}\frac{1}{(1+L)^2}\,du\right)ds\right]\\
& +\alpha z^2(t)\int_0^\infty\Big\{K(s)\frac{1}{(1+L)^2}\\
& \cdot\left[\int_{t-s}^{t}\left(r+\alpha\int_0^\infty K(x)\frac{1}{(1+L)^2}\,dx\right)\,dv\right]\Big\}ds\\
= & \Big\{-2r-\frac{2\alpha K_1}{(1+U)^2}+\left(\alpha r+\frac{\alpha^2K_1}{(1+L)^2}\right)\frac{K_2}{(1+L)^2}\\
& +\left(\alpha r+\frac{\alpha^2K_1}{(1+L)^2}\right)\frac{K_2}{(1+L)^2}\Big\}z^2(t)\\
= & -2\left\{r+\frac{\alpha K_1}{(1+U)^2}-\left(\alpha r+\frac{\alpha^2K_1}{(1+L)^2}\right)\frac{K_2}{(1+L)^2}\right\}z^2(t)\\
= & -2cz^2(t)
\end{aligned}
\tag{2.25}
$$

where

$$
c=r+\frac{\alpha K_1}{(1+U)^2}-\left(\alpha r+\frac{\alpha^2K_1}{(1+L)^2}\right)\frac{K_2}{(1+L)^2}>0 \text{ (by (2.15))}. \tag{2.26}
$$

Then

$$V(t) - V(0) = \int_0^t V'(s)\, ds \leq -2c \int_0^t z^2(s)\, ds.$$

Because $V(t)$ is a nonnegative function, we get

$$2c \int_0^t z^2(s)\, ds \leq V(0), \quad \text{for all} \quad t \geq 0$$

and thus

$$\int_0^\infty z^2(s)\, ds < \infty.$$

From Eq. (2.18) and $z(t)$ is bounded, we obtain that $|z'(t)|$ is also bounded.
Then $z^2(t)$ is uniformly continuous on $[0, \infty)$.
By Lemma 2.2, we obtain that

$$\lim_{t\to\infty} z^2(t) = 0$$

or, equivalently,

$$\lim_{t\to\infty} z(t) = 0.$$

This completes the proof of Theorem 2.3. □

REMARK 2.4. In Eq. (2.12), if the kernel function $K(t) = te^{-t}$ for all $t \geq 0$, then $K_1 = 1, K_2 = 2$ and $K_3 = 6$. Furthermore, if (2.15) holds then, by Theorem 2.3, the unique positive steady state p^* of (2.12)-(2.13) is globally asymtotical stable.

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