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EQUATIONS OF GEODESICS IN A TWO-DIMENSIONAL FINSLER SPACE WITH A GENERALIZED KROPINA METRIC

HONG-SUH PARK AND IL-YONG LEE

ABSTRACT. The geodesic equation in a two-dimensional Finsler space is given by the differential equation of the Weierstrass form. In the present paper, we express the differential equations of geodesics in a two-dimensional Finsler space with a generalized Kropina metric.

1. Introduction

The study on the differential equations of geodesics in a two-dimensional Finsler space $F^2 = (M^2, L)$ with an (α, β)-metric is interesting and useful. The geodesics of F^2 are regarded as the curves of an associated Riemannian space $R^2 = (M^2, \alpha)$ which are bent by the differential 1-form β. Recently, M. Matsumoto and the first author ([8]) have expressed the differential equations of the geodesics in two-dimensional Randers spaces and Kropina spaces in the most clear form $y'' = f(x, y, y')$.

The purpose of the present paper is devoted to studying the differential equations of geodesics in a two-dimensional Finsler space with a generalized Kropina metric and giving some examples.

2. Preliminaries

Let $F^2 = (M^2,\ L)$ be a two-dimensional Finsler space with a Finsler

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metric function $L(x^1,x^2;y^1,y^2)$. We denote $\partial f/\partial x^i = f_i, \quad \partial f/\partial y^i = f_{(i)}$, $(i=1,2)$ for any Finsler function $f(x^1,x^2;y^1,y^2)$. Hereafter, the suffices i,j run over 1, 2.

Since $L(x^1,x^2;y^1,y^2)$ is (1) p-homogeneous in (y^1,y^2), we have $L_{(j)(i)}y^i = 0$, which imply the existence of a function, so called the *Weierstrass invariant* $W(x^1,x^2;y^1,y^2)$ ([1], [6]) given by

$$\frac{L_{(1)(1)}}{(y^2)^2} = -\frac{L_{(1)(2)}}{y^1y^2} = \frac{L_{(2)(2)}}{(y^1)^2} = W(x^1,x^2;y^1,y^2). \tag{2.1}$$

In a two-dimensional associated Riemannian space $R^2 = (M^2,\ \alpha)$ with respect to $L=\alpha$ and $\alpha^2 = a_{ij}(x^1,x^2)y^iy^j$, the Weierstrass invariant W_r of R^2 is written as

$$W_r = \frac{1}{\alpha^3}\{a_{11}a_{22} - (a_{12})^2\}.$$

Further L_j are still (1)p-homogeneous in (y^1,y^2), so that we get

$$L_{j(i)}y^i = L_j. \tag{2.2}$$

The geodesic equations in F^2 along curve $C: x^i = x^i(t)$ are given by [1]

$$L_i - \frac{dL_{(i)}}{dt} = 0. \tag{2.3}$$

Substituting (2.2) in (2.3), we get

$$L_{1(2)} - L_{2(1)} + (y^1\dot{y}^2 - y^2\dot{y}^1)W = 0, \tag{2.4}$$

which is called the *Weierstrass form* of geodesic equation in F^2 ([6], [8]), where $\dot{y}^i = dy^i/dt$. For the metric function $L(x,y;\dot{x},\dot{y})$, (2.4) becomes to

$$\frac{\partial^2 L}{\partial\dot{y}\partial x} - \frac{\partial^2 L}{\partial\dot{x}\partial y} + (\dot{x}\ddot{y} - \dot{y}\ddot{x})\frac{\partial^2 L}{(\partial\dot{y})^2} = 0. \tag{2.5}$$

Let $\Gamma = \{\gamma_j{}^i{}_k(x^1,x^2)\}$ be the Levi-Civita connection of the associated Riemannian space R^2. We introduce the linear Finsler connection $\Gamma^* = (\gamma_j{}^i{}_k,\ \gamma_0{}^i{}_j,\ 0)$ and the h- and v-covariant differentiation in Γ^* are denoted by $(;i,\ (i))$ respectively, where the index (0) means the contraction with y^i. Then we have $y^i{}_{;j} = 0,\ \alpha_{;i} = 0$ and $\alpha_{(i);j} = 0$.

3. The geodesic equations with an (α, β)-metric

We consider a two-dimensional Finsler space $F^2 = (M^2,\ L(\alpha,\beta))$ with an (α,β)-metric, where $\beta = b_i(x^1,x^2)y^i$ ([1], [5]). For the metric function $L(\alpha,\beta)$, we have

$$(3.1) \qquad L_{;i} = L_\beta \beta_{;i}, \qquad L_{(i)} = L_\alpha \alpha_{(i)} + L_\beta b_i,$$

where $\alpha_{(i)} = a_{ir}y^r/\alpha$ and the subscriptions α, β of L are the partial derivatives of L with respect to α, β respectively. Then we have in Γ^*

$$L_{(j);i} = L_{(j)i} - L_{(j)(r)}\gamma_0{}^r{}_i - L_{(r)}\gamma_j{}^r{}_i,$$

from which

$$(3.2) \qquad L_{1(2)} - L_{2(1)} = L_{(2);1} - L_{(1);2} + L_{(2)(r)}\gamma_0{}^r{}_1 - L_{(1)(r)}\gamma_0{}^r{}_2.$$

From (2.1) and (3.2) we have

$$(3.3) \qquad L_{1(2)} - L_{2(1)} = L_{(2);1} - L_{(1);2} + (y^1\gamma_0{}^2{}_0 - y^2\gamma_0{}^1{}_0)W.$$

On the other hand, from (3.1) we have

$$(3.4) \qquad L_{(j);i} = L_{\alpha\beta}\beta_{;i}\alpha_{(j)} + L_{\beta\beta}\beta_{;i}b_j + L_\beta b_{j;i}.$$

Similarly to the case of $L(x^1,x^2;y^1,y^2)$ and $\alpha(x^1,x^2)$, we get the Weierstrass invariant $\omega(\alpha,\beta)$ for $L(\alpha,\beta)$ as follows:

$$(3.5) \qquad \omega = \frac{L_{\alpha\alpha}}{\beta^2} = -\frac{L_{\alpha\beta}}{\alpha\beta} = \frac{L_{\beta\beta}}{\alpha^2}.$$

Substituting (3.5) in (3.4), we have

$$(3.6) \qquad L_{(j);i} = \alpha\omega\beta_{;i}(\alpha b_j - \beta\alpha_{(j)}) + L_\beta b_{j;i}.$$

From (3.3) and (3.6) we have

$$(3.7) \qquad \begin{aligned} L_{1(2)} - L_{2(1)} =& \alpha\omega\{\beta_{;1}(\alpha b_2 - \beta\alpha_{(2)}) - \beta_{;2}(\alpha b_1 - \beta\alpha_{(1)})\} \\ & - L_\beta(b_{1;2} - b_{2;1}) + (y^1\gamma_0{}^2{}_0 - y^2\gamma_0{}^1{}_0)W. \end{aligned}$$

If we put ${y^i}_{;0} = \dot{y}^i + {\gamma_0}^i{}_0$, we get

$$y^1\dot{y}^2 - y^2\dot{y}^1 = y^1{y^2}_{;0} - y^2{y^1}_{;0} - (y^1{\gamma_0}^2{}_0 - y^2{\gamma_0}^1{}_0). \tag{3.8}$$

Substituting (3.7) and (3.8) in (2.4), we have

$$\begin{aligned} \alpha\omega\Big\{\beta_{;1}(\alpha b_2 - \beta\alpha_{(2)}) - \beta_{;2}(\alpha b_1 - \beta\alpha_{(1)})\Big\} \\ -L_\beta\left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right) + (y^1{y^2}_{;0} - y^2{y^1}_{;0})W = 0, \end{aligned} \tag{3.9}$$

where $\beta_{;i} = b_{r;i}y^r$. According to §2 of [4], the relation of $W,\ W_r$ and ω is written as follows:

$$W = (L_\alpha + \alpha\omega\gamma^2)W_r, \tag{3.10}$$

where $\gamma^2 = b^2\alpha^2 - \beta^2$ and $b^2 = a^{ij}b_ib_j$.
Therefore (3.9) is expressed as follows:

$$\begin{aligned} (L_\alpha + \alpha\omega\gamma^2)(y^1{y^2}_{;0} - y^2{y^1}_{;0})W_r - L_\beta\left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right) \\ +\alpha\omega\Big\{b_{0;1}(\alpha b_2 - \beta\alpha_{(2)}) - b_{0;2}(\alpha b_1 - \beta\alpha_{(1)})\Big\} = 0. \end{aligned} \tag{3.11}$$

Thus we have the following

THEOREM 3.1. *In a two-dimensional Finsler space F^2 with an (α, β)-metric, the differential equation of a geodesic is given by* (3.11).

Suppose that α be positive-definite. Then we may refer to an *isothermal coordinate system* $(x^i) = (x, y)$ ([3]) such that

$$\alpha = aE, \quad a = a(x,y) > 0, \quad E = \sqrt{\dot{x}^2 + \dot{y}^2},$$

that is, $a_{11} = a_{22} = a^2$, $a_{12} = 0$ and $(y^1, y^2) = (\dot{x}, \dot{y})$. From $\alpha^2 = a_{ij}(x)y^iy^j$ we get $\alpha\alpha_{(i)(j)} = a_{ij} - a_{ir}a_{js}y^ry^s/\alpha^2$. Therefore we have

$\alpha\alpha_{(1)(1)} = (a\dot{y}/E)^2$ and $W_r = a/E^3$. Furthermore the Christoffel symbols are given by

$$\gamma_1{}^1{}_1 = -\gamma_2{}^1{}_2 = \gamma_1{}^2{}_2 = \frac{a_x}{a}, \qquad \gamma_1{}^1{}_2 = -\gamma_1{}^2{}_1 = \gamma_2{}^2{}_2 = \frac{a_y}{a},$$

where $a_x = \partial a/\partial x,\ a_y = \partial a/\partial y$. Therefore we have

$$(3.12) \qquad (y^1 y^2{}_{;0} - y^2 y^1{}_{;0})W_r = \frac{a}{E^3}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \frac{1}{E}(a_x\dot{y} - a_y\dot{x}).$$

Next, calculating $\gamma^2 = b^2\alpha^2 - \beta^2,\quad b_{0;1}(\alpha b_2 - \beta\alpha_{(2)})$ and $b_{0;2}(\alpha b_1 - \beta\alpha_{(1)})$, we have

$$(3.13) \quad \gamma^2 = \{(b_1)^2 + (b_2)^2\}(\dot{x}^2 + \dot{y}^2) - (b_1\dot{x} + b_2\dot{y})^2 = (b_1\dot{y} - b_2\dot{x})^2,$$

$$(3.14) \qquad b_{r;1}(\alpha b_2 - \beta\alpha_{(2)})y^r = \frac{a}{E}b_{0;1}(b_2\dot{y} - b_1\dot{x})\dot{x},$$

$$(3.15) \qquad b_{r;2}(\alpha b_1 - \beta\alpha_{(1)})y^r = \frac{a}{E}b_{0;2}(b_1\dot{y} - b_2\dot{x})\dot{y}.$$

Substituting (3.12), (3.13), (3.14) and (3.15) in (3.11), we have

$$(3.16) \qquad \begin{aligned} &\Big\{L_\alpha + aE\omega(b_1\dot{y} - b_2\dot{x})^2\Big\}\Big\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2(a_x\dot{y} - a_y\dot{x})\Big\} \\ &\qquad - E^3 L_\beta(b_{1y} - b_{2x}) - E^3 a^2\omega(b_1\dot{y} - b_2\dot{x})b_{0;0} = 0, \end{aligned}$$

where

$$(3.17) \qquad \begin{aligned} b_{0;0} = b_{r;s}y^r y^s &= (b_{1x}\dot{x} + b_{1y}\dot{y})\dot{x} + (b_{2x}\dot{x} + b_{2y}\dot{y})\dot{y} \\ &+ \frac{1}{a}\Big\{(\dot{x}^2 + \dot{y}^2)(a_x b_1 + a_y b_2) - 2(b_1\dot{x} + b_2\dot{y})(a_x\dot{x} + a_y\dot{y})\Big\}, \end{aligned}$$

where $b_{ix} = \partial b_i/\partial x$ and $b_{iy} = \partial b_i/\partial y$. Thus we have the following

THEOREM 3.2. *In a two-dimensional Finsler space F^2 with an (α,β)-metric, if we refer to an isothermal coordinate system (x, y) such that $\alpha = aE$, then the differential equation of a geodesic is given by* (3.16) *and* (3.17).

4. Geodesics in the Finsler space with a generalized Kropina metric

The (α,β)-metric $L(\alpha,\beta)=\alpha^{m+1}\beta^{-m}$ $(m\neq 0,-1)$ is called a *generalized Kropina metric* ([5]). We consider a two-dimensional Finsler space with a generalized Kropina metric in this section. Then

$$(4.1)\quad L_\alpha=\frac{(m+1)\alpha^m}{\beta^m},\quad L_\beta=-\frac{m\alpha^{m+1}}{\beta^{m+1}},\quad \omega=\frac{m(m+1)\alpha^{m-1}}{\beta^{m+2}}.$$

Substituting (4.1) in (3.16), we obtain the differential equation of a geodesic in an isothermal coordinate system (x,y) with respect to α as follows:

(4.2)

$$(m+1)\Big\{\beta^2+m(b_1\dot y-b_2\dot x)^2\Big\}\Big\{a(\dot x\ddot y-\dot y\ddot x)+E^2(a_x\dot y-a_y\dot x)\Big\}$$
$$+maE^2\Big\{E^2\beta(b_{1y}-b_{2x})-(m+1)(b_1\dot y-b_2\dot x)b_{0;0}\Big\}=0.$$

If the parameter t of curve C is chosen x of (x,y), then $\dot x=1,\ \dot y=y',\ \ddot x=0,\ \ddot y=y'',\ E^2=1+(y')^2$. Therefore (4.2) is written in the form

(4.3)

$$\Big\{(b_1)^2+m(b_2)^2-2(m-1)b_1b_2y'+\{m(b_1)^2+(b_2)^2\}(y')^2\Big\}$$
$$\Big\{y''+\frac{1}{a}\{1+(y')^2\}(a_xy'-a_y)\Big\}$$
$$+\frac{m}{(m+1)}(1+(y')^2)\Big\{\{1+(y')^2\}(b_1+b_2y')(b_{1y}-b_{2x})$$
$$-(m+1)(b_1y'-b_2)b^*_{0;0}\Big\}=0,$$

where

$$(4.4)\quad \begin{aligned} b^*_{0;0}=&(b_{1x}+b_{1y}y')+(b_{2x}+b_{2y}y')y'\\ &+\frac{1}{a}\Big\{\{1+(y')^2\}(a_xb_1+a_yb_2)-2(b_1+b_2y')(a_x+a_yy')\Big\}.\end{aligned}$$

It seems quite complicated form, but y'' is given as a fractional expression in y'.

Thus we have the following

THEOREM 4.1. *Let F^2 be a two-dimensional space with a generalized Kropina metric. If we refer to a local coordinate system (x, y) with respect to α, then the differential equation of a geodesic $y = y(x)$ of F^2 is the form*

$$y'' = \frac{g(x, y, y')}{f(x, y, y')},$$

where $f(x, y, y')$ is a quadratic polynomial in y' and $g(x, y, y')$ is a polynomial in y' of degree at most five.

In order to find a concrete form, we treat the case of which the associated Riemannian space is Euclidean with orthonormal coordinate system. Then $a = 1$ and $a_x = a_y = 0$. If we take a scalar function b such that $b_1 = b_x, b_2 = b_y$, then $b_{1y} - b_{2x} = 0$. Therefore (4.3) is reduced to

$$(4.5)\quad y'' = \frac{m\{(1 + (y')^2\}(b_x y' - b_y)\{b_{xx} + 2b_{xy}y' + b_{yy}(y')^2\}}{(b_x)^2 + m(b_y)^2 - 2(m - 1)b_x b_y y' + \{m(b_x)^2 + (b_y)^2\}(y')^2}.$$

Thus we have the following

COROLLARY 4.2. *Let F^2 be a two-dimensional Finsler space with a generalized Kropina metric. If we refer to an orthonormal coordinate system (x, y) with respect to α and $b_1 = \partial b/\partial x$, $b_2 = \partial b/\partial y$ for a scalar b, then the differential equation of a geodesic $y = y(x)$ of F^2 is given by* (4.5).

5. Examples

EXAMPLE 1. In the Finsler space with an (α, β)-metric, the special (α, β)-metric L satisfying $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$ (c_1, c_2, c_3 are constants) was introduced in [9] as the generalization of the Randers metric. The (α, β)-metric L satisfying $L^2 = 2\alpha\beta$ is the case of

$c_1 = c_3 = 0, c_2 = 1$ in the above special (α, β)-metric. This metric is also considered as a generalized $(-1/2)$-Kropina metric.

In a two-dimensional Finsler space with an (α, β)-metric L satisfying $L^2 = 2\alpha\beta$,

$$L_\alpha = \frac{\beta}{L}, \qquad L_\beta = \frac{\alpha}{L}, \qquad \omega = -\frac{1}{L^3}. \tag{5.1}$$

Substituting (5.1) in (3.16), we obtain the differential equation of a geodesic as follows:

$$\begin{aligned} &\left\{2\beta^2 - (b_1\dot{y} - b_2\dot{x})^2\right\}\left\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2(a_x\dot{y} - a_y\dot{x})\right\} \\ &\qquad -2aE^4\beta(b_{1y} - b_{2x}) + aE^2(b_1\dot{y} - b_2\dot{x})b_{0;0} = 0. \end{aligned} \tag{5.2}$$

If x of (x, y) is chosen as the parameter t, then $\dot{x} = 1,\ \dot{y} = y',\ \ddot{x} = 0,\ \ddot{y} = y''$ and $E^2 = 1 + (y')^2$. Therefore (5.2) is reduced to

(5.3)

$$\begin{aligned} &\left\{2(b_1)^2 - (b_2)^2 + 6b_1b_2y' + \{2(b_2)^2 - (b_1)^2\}(y')^2\right\}\left\{ay'' + \{1 + (y')^2\}\right. \\ &\qquad \left.(a_xy' - a_y)\right\} - 2a\{1 + (y')^2\}^2(b_1 + b_2y')(b_{1y} - b_{2x}) \\ &\qquad + a\{1 + (y')^2\}(b_1y' - b_2)b^*_{0;0} = 0, \end{aligned}$$

where

$$\begin{aligned} b^*_{0;0} &= (b_{1x} + b_{1y}y') + (b_{2x} + b_{2y}y')y' \\ &\quad + \frac{(1 + (y')^2)}{a}(a_xb_1 + a_yb_2) - \frac{2(b_1 + b_2y')}{a}(a_x + a_yy'). \end{aligned} \tag{5.4}$$

If we refer to a local coordinate system (x, y) with respect to the associated Riemannian space which is Euclidean with an orthonormal coordinate system, then $a = 1, a_x = a_y = 0$, so that (5.3) is reduced to

$$\begin{aligned} &y''\left\{2(b_1)^2 - (b_2)^2 + 6b_1b_2y' + \{2(b_2)^2 - (b_1)^2\}(y')^2\right\} \\ &\qquad -2\{1 + (y')^2\}^2(b_1 + b_2y')(b_{1y} - b_{2x}) \\ &\qquad +\{1 + (y')^2\}(b_1y' - b_2)b^*_{0;0} = 0, \end{aligned} \tag{5.5}$$

where

$$b^*_{0;0} = (b_{1x} + b_{1y}y') + (b_{2x} + b_{2y}y')y'. \tag{5.6}$$

If we take b_1 and b_2 such that $b_1 = b_x$ and $b_2 = b_y$ for a scalar b, then $b_{1y} - b_{2x} = 0$. From (5.5) and (5.6) we have the following

THEOREM 5.1. *Let F^2 be a two-dimensional Finsler space with an (α, β)-metric L satisfying $L^2 = 2\alpha\beta$. If we refer to an orthonormal coordinate system (x, y) with respect to α and $b_1 = \partial b/\partial x$, $b_2 = \partial b/\partial y$ for a scalar b, then the differential equation of a geodesic $y = y(x)$ of F^2 is given by*

$$y'' = \frac{C_0 + C_1 y' + C_2(y')^2 + C_3(y')^3 + C_4(y')^4 + C_5(y')^5}{2(b_x)^2 - (b_y)^2 + 6b_x b_y y' + \{2(b_y)^2 - (b_x)^2\}(y')^2},$$

where

$$C_0 = -b_y b_{xx}, \quad C_1 = b_x b_{xx} - 2b_y b_{xy}, \quad C_2 = 2b_x b_{xy} - b_y(b_{xx} + b_{yy}),$$
$$C_3 = b_x(b_{xx} + b_{yy}) - 2b_y b_{xy}, \quad C_4 = 2b_x b_{xy} - b_y b_{yy}, \quad C_5 = b_x b_{yy}.$$

EXAMPLE 2. If $m = 1$ in a generalized Kropina metric, then $L = \alpha^2\beta^{-1}$, that is, L is the Kropina metric. Therefore if we substitute $m = 1$ in (4.3), then y'' of the geodesic equations $y = y(x)$ is written as a polynomial in y' of degree at most three, that is,

$$y'' = A_0 + A_1 y' + A_2(y')^2 + A_3(y')^3, \tag{5.7}$$

where $A_\sigma(\sigma = 0, 1, 2, 3)$ are functions of (x, y). That is the same result which is obtained in [8].

On the other hand, S.Bácsó and M. Matsumoto ([2]) proved as follows: *a two-dimensional Finsler space is a Douglas space if and only if, in a local coordinate system (x, y), y'' of the geodesic equations $y = y(x)$ is a polynomial in y' of degree at most three.*

Therefore, from (5.7) we have the following

THEOREM 5.2. *A two-dimensional Kropina space in a local coordinate system (x, y) of the associated Riemannian space is a Douglas space.*

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HONG-SUH PARK, DEPARTMENT OF MATHEMATICS, YEUNGNAM UNIVERSITY, GYONGSAN 712-749, KOREA

IL-YONG LEE, DEPARTMENT OF MATHEMATICS, KYUNGSUNG UNIVERSITY, PUSAN 608-736, KOREA,
E-mail: iylee@star.kyungsung.ac.kr