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CONTROLLABILITY OF LINEAR AND SEMILINEAR CONTROL SYSTEMS

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ABSTRACT. Our purpose is to seek that the reachable set of the semilinear system $\frac{d}{dt}x(t) = Ax(t) + f(t, x(t)) + Bu(t)$ is equivalent to that of its corresponding to linear system(the case where $f = 0$). Under the assumption that the system of generalized eigenspaces of A is complete, we will show that the reachable set corresponding to the linear system is independent of t in case A generates C_0-semigroup. An illustrative example for retarded system with time delay is given in the last section.

1. Introduction

Suppose that we are given a semilinear control system described by

$$\tag{1.1} \begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t > 0, \\ x(0) = g \end{cases}$$

on a reflexive Banach space X, where A generates a C_0-semigroup $S(t)$. The controller operator B is bounded linear operator from another Banach space U to X and $u \in C^1([0, T); U)$. Here $C^1([0, T); U)$ denotes the set of all continuously differentiable functions from $[0, T)$ into U with the norm defined by

$$||u||_{C^1} = \sup\{u(s) + u'(s) : s \in [0, T)\}$$

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where $C^1 = C^1([0,T);U)$ for the sake of simplicity.

For each $t > 0$, we define reachable sets of classical solutions of the initial value problem (1.1) by

$$L_t = \{y \in X : y = \int_0^t S(t-s)Bu(s)ds, \quad u \in C^1([0,t),U)\}$$
$$R_t = \{y \in X : y = \int_0^t S(t-s)(f(s,x(s)) + Bu(s))ds,$$
$$u \in C^1([0,t),U)\}.$$

Our purpose is to seek that the reachable set R_t of the semilinear system (1.1) is equivalent to that of its corresponding to linear system(the case where $f \equiv 0$). It is seen that the relation $L_{t'} \subset L_t$ if $t < t'$, but the converse relation may be not true if the semigroup $S(t)$ generated by A is not analytic. Under the assumption the system of generalized eigenspaces of A is complete, we will show that L_t is independent of the time t. But it is well known that it also holds under admissible control set contracted in $L^2(0,t;X)$. If A generates an analytic semigroup, we know that reachable set of linear system is invariable. For the control problem (1.1) where the operator A generates an analytic semigroup was dealt with in M. Yamamoto and J. Y. Park [15]. But our case of the equation (1.1) is that the operator A is the infinitesimal generator of C_0-semigroup. Now we note that it is known that the C_0semigroup generated by A associated with the equation (1.1) is not compact operator in general (see theorem 5.3 in [3]). The control problem of (1.1) where the semigroup generated by A is compact operator was obtained by K. Naito [6,7] using topological degree theory. An illustrative example for retarded system with time delay is given in the last section.

2. Reachable sets of initial value problem

For system (1.1), let f be a nonlinear mapping from $\mathcal{R} \times X$ into X. We assume that

(F). $f : [0,T] \times X \to X$ is uniformly Lipschitz in X and for each $x \in X$, $f(\cdot\,, x)$ is continuous from $[0,T]$ to X.

Thus, for any x_1, $x_2 \in X$ there exists a constant $L > 0$ such that

$$||f(t, x_1) - f(t, x_2)|| \leq L||x_1 - x_2|| \tag{2.1}$$

where $||\cdot||$ denotes norm of X. If initial data $x(0) = g \in D(A)$ and $u \in C^1([0,T); U)$ for fixed time T then the initial value problem (1.1) has a unique classical solution on $[0,T)$(e.g. Pazy (Ref 9, Theorem 6.1.7)). The solution of initial value problem (1,1) is the following form:

$$x(t; g, f, u) = S(t)g + \int_0^t S(t-s)f(s, x(s))ds + \int_0^t S(t-s)Bu(s)ds.$$

For the sake of simplicity we assume that $S(t)$ is uniformly bounded; there exists a constant $M \geq 1$ such that

$$||S(t)|| \leq M. \tag{2.2}$$

For $T > 0$, $g \in D(A)$ and $u \in C^1([0,T); U)$ we set

$$\begin{aligned}
L_T(g) &= \{x(T; g, 0, u) : u \in C^1([0,T); U)\},\\
R_T(g) &= \{x(T; g, f, u) : u \in C^1([0,T); U)\},\\
L(g) &= \bigcup_{T>0} L_T(g), \quad R(g) = \bigcup_{T>0} R_T(g),\\
L_T^k(g) &= \{x(T; g, 0, u) : ||u||_{C^1([0,T);U)} \leq k\},\\
R_T^k(g) &= \{x(T; g, f, u) : ||u||_{C^1([0,T);U)} \leq k\}.
\end{aligned}$$

Since X is a reflexive space, we can consider the dual system of the problem (1.1). Moreover, we introduce the unobservable subspace for the dual system of the problem (1.1):

$$N_T = \bigcap_{0\leq t\leq T} \text{Ker } B^*S^*(t), \quad N = \bigcap_{T>0} N_T.$$

Here, B^* and $S^*(t)$ are dual operators of B and $S(t)$, respectively.

LEMMA 2.1. *For* $0 \le t < t'$ *we have that* $L_{t'}(0) \subset L_t(0)$.

Proof. (1) Let $t < t'$ and suppose $y = \int_0^{t'} S(t' - s)Bu(s)ds \in L_{t'}(0)$. Put

$$v(s) = u(s + t' - t) + \frac{1}{t}S(s)\int_0^{t'-t} S(t' - t - r)u(r)dr.$$

Then since $\int_0^{t'-t} S(t' - t - r)u(r)dr \in D(A)$ in terms of (2.5) in Theorem 1.2.4 of [9] and $u \in C^1([0, t'); U)$ we know that $v \in C^1([0, t); U)$ and $y = \int_0^t S(t - s)v(s)ds \in L_t(0)$. □

LEMMA 2.2. *Let* $x_u(t) = x(t; g, f, u)$. *Then for* $0 < t < T$ *there exists a constant* C *such that*

$$||x_u(t)|| \le C(||g|| + ||u||_{C^1})$$
$$||f(t, x_u(t))|| \le C(||g|| + ||u||_{C^1})$$

and

$$||f(t, x_u(t)) - f(t, x_v(t))|| \le C||u - v||_{C^1}$$

for every $u, v \in C^1([0, T); U)$.

Proof. From the form of solution of (1.1) we have

$$\begin{aligned}||x(t)|| &\le M||g|| + M\int_0^t ||f(s, x(s))||ds + M||B||\int_0^t ||u||_{C^1}ds \\ &\le M||g|| + TM||B||||u||_{C^1} + ML\int_0^t ||x(s)||ds.\end{aligned}$$

By virtue of Gronwall's inequality we have the first inequality. From (2.1), (2.2) and the inequality above, it follows the second and the third inequalities. □

So far we have know that $L_T(0)$ may not be independent of the time T if $u \in C^1([0, T); U)$. So we proceed the investigation of reachable set by using the spectral theory.

Let $\sigma_p(A)$ be the point spectrum of A and $\sigma_0(A)$ be the set of all poles of $(\mu - A)^{-1}$. Let λ be a pole of the resolvent of A of order k_λ and P_λ be the spectral projection

$$P_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (\mu - A)^{-1} d\mu$$

where Γ_λ is a closed rectifiable curve containing λ such that it surrounds no point $\sigma(A)$ except λ. Then the generalized eigenspace corresponding to λ is given by

$$X_\lambda \equiv P_\lambda X \equiv \operatorname{Ker}(\lambda I - A)^{k_\lambda}$$

where the natural number k_λ is the order of a pole λ of $(\mu I - A)^{-1}$. Defining the operator Q_λ by

$$Q_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (\mu - \lambda)(\mu - A)^{-1} d\mu,$$

we have that $Q_\lambda^{k_\lambda} \equiv 0$, Im $Q_\lambda \subset X_\lambda$, and $(A - \lambda)^n P_\lambda = Q_\lambda^n$.

DEFINITION 2.1. (the case linear system, that is, $f \equiv 0$) For any $\lambda \in \sigma_o(A)$,
(1) the linear system of (1.1) is said to be X_λ-controllable (resp. in time T) for initial value g if $X_\lambda \subset \overline{L(g)}$ (resp. $X_\lambda \subset \overline{L_T(g)}$).
(2) The dual linear system of (1.1) is said to be $X^*_{\overline{\lambda}}$-observable (resp. in time T) if $N \cap X^*_{\overline{\lambda}} = \{0\}$(resp. $N_T \cap X^*_{\overline{\lambda}} = \{0\}$), where $X^*_{\overline{\lambda}}$ is the generalized eigenspace for $\overline{\lambda}$ which is an eigenvalue of A^*.

We can also define approximate controllability for the semilinear system in case where $f \neq 0$ by replacing the reachable set L_T by R_T. The following Lemma follows from Theorem 1 of [10].

LEMMA 2.3. *Let λ be a pole of the resolvent of A of order k_λ. Then the following statements are equivalent:*
(1) *The linear system* (1.1) *is X_λ-controllable for initial value $g = 0$ in X.*
(2) *The dual linear system of* (1.1) *is $X^*_{\overline{\lambda}}$- observable in X^*.*
(3) $(\bigcap_{j=0}^{k_\lambda - 1} \operatorname{Ker}\ B^*(Q^*_{\overline{\lambda}})^j) \cap X^*_{\overline{\lambda}} = \{0\}$.
(4) $\operatorname{Cl}(\operatorname{Span}\{Q_\lambda^j Bu : 0 \leq j \leq k_\lambda - 1,\ \ t \geq 0,\ \ u \in C^1([0,t);U)\}) = X_\lambda$, *where* Cl *denotes the closure in X.*

LEMMA 2.4. *The following statements are equivalent:*
(1) *The linear system* (1.1) *is* X_λ*-controllable in time* T *in* X.
(2) *The linear system* (1.1) *is* X_λ*-controllable in* X.
(3) *The linear system* (1.1) *is* $X^*_{\overline{\lambda}}$*-observable in time* T *in* X^*.
(4) *The linear system* (1.1) *is* $X^*_{\overline{\lambda}}$*-observable in* X^*.

Proof. By Lemma 2.3 it is sufficient to prove that (1) $\Leftrightarrow$ (2). Using the dual theorem we obtain that

$$\overline{L_T(g)}^{\perp} = N_T = \bigcap_{0\leq t<T} \text{Ker } B^*S^*(t).$$

Let $f \in X^*_{\overline{\lambda}}$. Then we have

$$S^*(t)f = e^{\overline{\lambda}t} \sum_{n=0}^{k_\lambda-1} \frac{t^n}{n!}(Q^*_{\overline{\lambda}})^n f$$

and, hence $f \in \bigcap_{0\leq t<T} \text{Ker } B^*S^*(t)$ if and only

$$e^{\overline{\lambda}t} \sum_{n=0}^{k_\lambda-1} \frac{t^n}{n!} B^*(Q^*_{\overline{\lambda}})^n f = 0$$

for $t \in [0,T)$. Since the function $t \longmapsto ct^n e^{\overline{\lambda}t}$ is entire, the above condition vanishes in a given interval $[0,T)$ if and only if it vanishes identically. So it is easily seen that the condition (1) holds if and only if the condition (3) of Lemma 2.3 is satisfied. Hence, from Lemma 2.1 the proof is complete. □

DEFINITION 2.2. The system of generalized eigenspaces of A is complete in X if

$$\text{Cl}(\bigcup_{\lambda\in\sigma_o(A)} X_\lambda) = X.$$

DEFINITION 2.3. (1) The linear system of (1.1) is said to be X-approximately controllable for initial value g (resp. in time T) if $\overline{L(g)} = X$ (resp. $\overline{L_T(g)} = X$).
(2) The dual linear system of (1.1) is X^*-controllable (resp. in time T) if $N = \{0\}$ (resp. $N_T = \{0\}$).

We assume that
(H) $\sigma_p(A) = \sigma_0(A)$ and the system of generalized eigenspaces of A is complete in X.

THEOREM 2.2. *Under the condition* (H) *we have* $\overline{L_T(g)} = \overline{L(g)}$ *is independent of the time* T.

The proof is immediately shown from Lemma 2.4.

REMARK 2.1. *If the admissible control set is contracted by in* $L^2(0, T; U)$ *and the controller operator* B *is bounded then it is known that* $L_t(0)$ *is independent of the time* t *as is seen in* [13; Lemma 7.4.1].

3. Approximate controllability

First of all, we will show that X-approximate controllability for the semilinear system (1.1) implies the linear system (1.1) in case where $f \equiv 0$ is X-approximately controllable.

THEOREM 3.1. *For any* $k > 0$ *there exists a constant* $\alpha > 0$ *such that*

$$\overline{R_T^k(0)} \subset \alpha \overline{L_T^k(0)}.$$

Proof. Let $z_0 \notin \overline{L_T^k(0)}$. Since $\overline{L_T^k(0)}$ is a balanced closed convex subset, we have

$$\inf\{||z_0 - z|| : z \in \overline{L_T^k(0)}\} = d.$$

obtain that for every $u \in C^1([0,T);X)$ satisfying $||u||_{C^1} \leq k$,

$$\begin{aligned} &||x(T;0,f,u) - \alpha z_0|| \\ &\geq ||\int_0^T S(T-s)Bu(s)ds - \alpha z_0|| - ||\int_0^T S(T-s)f(s,x(s))ds|| \\ &\geq \alpha d - kMCT > 0. \end{aligned}$$

Hence the proof is complete. □

To prove the converse relation of Theorem 3.1, we need the following hypothesis:
(F1) $|f(t,x)| \leq L$, $x \in X$.
We also assume that
(H1) $S(t)(D(A)) \subset \overline{L_t(0)}$ for some time t.

This assumption (H1) is more general case than the known several conditions in [6,7,15].

THEOREM 3.2. *Under conditions* (F1), (H) *and* (H1), *we have that for any* $g \in D(A)$ *and* $T > 0$

$$\overline{L_T(g)} \subset \overline{R_T(g)}.$$

Proof. Let $\epsilon > 0$ and $z \in \overline{L_T(g)}$ and let $\delta < \epsilon/3(ML)^{-1}$. Put $x_0(s) = x(s;g,f,0)$ and $x_1 = x(T-\delta;g,f,0)$, where $x(T-\delta;g,f,0) = S(T-\delta)g + \int_0^{T-\delta} S(T-\delta-s)f(s,x_0(s))ds$. Then we have that $S(\delta)x_1 \in \overline{L_T(0)}$ in view of assumption (H1) and Theorem 2.2. Consider the following problem:

$$\begin{aligned} &\frac{d}{dt}y(t) = Ay(t) + Bu(t), \quad T-\delta < s < T, \\ &y(T-\delta) = x_1. \end{aligned}$$

Then since the form of a solution of above equation is $y_u(T) = S(\delta)x_1 + \int_{T-\delta}^T S(T-s)Bu(s)ds$ we see that $y_u(T) \in \overline{L_T(0)}$ from Theorem 2.2. Therefore there exists $u_1 \in C^1([T-\delta,T);U)$ such that

$$||y_{u_1}(T) - z|| < \frac{\epsilon}{3}. \tag{3.1}$$

Now we set

$$v(s) = \begin{cases} 0 & \text{if } 0 \le s \le T - \delta, \\ u_1(s) & \text{if } T - \delta < s < T. \end{cases}$$

Then $v \in L^2([0,T);U)$. Since $C_0^\infty([0,T);X)$ is dense in $L^2([0,T);X)$, also $C^1([0,T);X)$ is dense in $L^2([0,T);X)$, there exists $u_2 \in C^1([0,T);X)$ such that

$$||v - u_2||_{C^1} \le \frac{\epsilon}{3}\{TM||B|| + TMC\}. \tag{3.2}$$

Observing that

$$x(T;g,f,v) = S(\delta)x_1 + \int_{T-\delta}^{T} S(T-s)(f(s,x_{u_1}(s) + Bu_1(s))ds,$$

from (3.1) and (3.2) we obtain that

$$\begin{aligned}
||x(T;g,f,u_2) - z|| &\le ||x(T;g,f,v) - z|| \\
&\quad + ||x(T;g,f,u_2) - x(T;g,f,v)|| \\
&\le ||S(\delta)x_1 + \int_{T-\delta}^{T} S(T-s)Bu_1(s)ds - z|| \\
&\quad + ||\int_0^T S(T-s)B(u_2(s) - v(s))ds|| \\
&\quad + ||\int_0^T S(T-s)(f(s,x_{u_2}(s)) - f(s,x_v(s)))ds \\
&\quad + ||\int_{T-\delta}^{T} S(T-s)f(s,(x_{u_1}(s))ds|| \\
&\le \frac{\epsilon}{3} + (TM||B|| + TMC)||u_1 - u_2||_{C^1} + \delta ML \\
&< \epsilon.
\end{aligned}$$

Hence the proof is complete. □

COROLLARY 3.1. *Let us assume the hypotheses* (H), (H1) *and* (F). *Then* $\overline{L_T(g)} = \overline{R_T(g)} = X$ *for any* $g \in D(A)$ *and* $T > 0$. *Therefore, if the linear system* (1.1) *is approximately controllable, then so is the nonlinear system* (1.1).

The proof of this Corollary holds from Theorem 3.1 and 3.2.

4. Application for retarded control system

Let Ω be a bounded domain in $\mathcal{R}^n$ with smooth boundary $\partial\Omega$. Consider an elliptic differential operator of second order

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \overline{\Omega}$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$. For $1 < p < \infty$ we denote the realization of $\mathcal{A}$ in $L^p(\Omega)$ under the Dirichlet boundary condition by A_p:

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$
$$A_p u = \mathcal{A}u \quad \text{for} \quad u \in D(A_p).$$

It is known that $-A_p$ generates analytic semigroups in $L^p(\Omega)$. For an interpolation couple of Banach spaces X_0 and X_1, $(X_0, X_1)_{\theta,p}$ and $[X_0, X_1]_\theta$ denote the real and complex interpolation spaces between X_0 and X_1, respectively. From the result of R. Seeley [10] we obtain that

$$[D(A_p), L^p(\Omega)]_{\frac{1}{2}} = W_0^{1,p}(\Omega)$$

and hence, may consider that

$$D(A_p) \subset W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,p}(\Omega).$$

Then by the interpolation theory, the operator A_p can be extended to an operator from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$, where $W^{-1,p}(\Omega)$ is the dual space of $W_0^{1,p'}(\Omega)$, $p' = p/(p-1)$, which is also denoted by A_p. Furthermore, as seen in proposition 3.1 in [5], it is known that A_p generates an analytic semigroup in $W^{-1,p}(\Omega)$.

Hence we intend to investigate the control problem for following retarded system:

$$
\begin{aligned}
(4.1) \qquad \frac{d}{dt}x(t) =& A_2x(t) + A_2x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\
& + f(t, x(t)) + B_0u(t), \quad t \in (0, T] \\
(4.2) \qquad x(0) =& g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0),
\end{aligned}
$$

in the space $W^{-1,2}(\Omega)$. Here, $A_2u = -\mathcal{A}(x, D_x)u$. The function $a(\cdot)$ is assume to be a real valued Hölder continuous in $[-h, 0]$ and the controller operator B_0 is a bounded linear operator from some Banach space U to $L^2(\Omega)$. The nonlinear operator f is defined by as follows;

$$
f(t, x) = \frac{\sigma x}{1 + ||x||}, \quad \sigma > 0.
$$

Then the Lipschitz continuity of f and the condition (F) in section 2 are satisfied. The initial data g^0, g^1 are given functions so that needed for the construction of solution semigroup for (4.1) and (4.2) and of $W^{-1,2}(\Omega)$- valued controller. From the relation above we shall deal with approximate controllability and observability for the system (4.1) and (4.2) in the space $W^{-1,2}(\Omega)$. It is also known that the fundamental solution $W(t)$ is bounded and differentiable $t \neq nr$, $n = 0,\ 1,\ 2,\ ...$ as seen in [14]. The fundamental solution $W(t)$ of the equation (4.1) and (4.2) is defined as follows:

$$
\begin{aligned}
& \frac{d}{dt}W(t) = A_2W(t) + A_2W(t-h) + \int_{-h}^{0} a(s)A_2W(t+s)ds, \ t > 0, \\
& W(0) = I, \quad W(s) = 0, \ s \in [-h, 0).
\end{aligned}
$$

Since we assumed that $a(\cdot)$ is Hölder continuous, as seen in [14] the fundamental solution exists. The solution of (4.1) and (4.2) is expressed by

$$x(t) = W(t)g^0 + \int_{-h}^{0} U_t(s)g^1(s)ds + \int_0^t W(t - x(\tau))f(\tau, x(\tau))d\tau,$$

$$U_t(s) = W(t - s - h)A_2 + \int_{-h}^{s} W(t - s - \sigma)a(\sigma)A_2 d\sigma$$

(cf. S. Nakagiri [8]).

THEOREM 4.1. *Let H and V be Hilbert spaces such that $(D(A), H)_{\frac{1}{2},2}$ satisfying*

$$||u||_V \leq C_1 ||u||^{\frac{1}{2}}_{D(A_2)} ||u||^{\frac{1}{2}}_H \tag{4.3}$$

for some constant $C_1 > 0$. Then there exists a unique solution x of (4.1) and (4.2) such that $x \in L^2(0, T; V)$ for any $g = (g^0, g^1) \in Z \equiv H \times L^2(-h, 0; V)$.

Proof. Let $f \in L^2(0, T; H)$ and $x(t) = \int_0^t W(t - s)f(s)ds$. Then there exists a constant C such that

$$||x||_{L^2(0,T;D(A_2))} \leq C||f||_{L^2(0,T;H)} \tag{4.4}$$

(cf. see [2]). Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t W(t-s)f(s)ds|^2 dt \\ &\leq C \int_0^T (\int_0^t |f(s)|ds)^2 dt \\ &\leq C \int_0^T t \int_0^t |f(s)|^2 ds\, dt \\ &\leq C\frac{T^2}{2} \int_0^T |f(s)|^2 ds, \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \leq CT||f||_{L^2(0,T;H)}. \tag{4.5}$$

From (4.3), (4.4), and (4.5) it holds that

$$(4.6) \qquad ||x||_{L^2(0,T;V)} \leq C\sqrt{T}||f||_{L^2(0,T;H)}.$$

Put

$$J(x)(t) = W(t)g^0 + \int_{-h}^{0} U_t(s)g^1(s)ds + \int_0^t W(t-\tau)f(\tau, x(\tau))d\tau.$$

Then from (4.6) and

$$J(x_1)(t) - J(x_2)(t) = \int_0^t W(t-s)(f(s, x_1(s) - f(s, x_2(s)))ds$$

we have

$$\begin{aligned} ||J(x_1) - J(x_2)||_{L^2(0,T;V)} &\leq C\sqrt{T}||f(\cdot, x_1(\cdot)) - f(\cdot, x_2(\cdot))||_{L^2(0,T;H)} \\ &\leq CL\sqrt{T}||x_1(\cdot) - x_2(\cdot)||_{L^2(0,T;V)}. \end{aligned}$$

Let us fix the time T such that $CL\sqrt{T} < 1$. By the contraction mapping the solution of (4.1) and (4.2) exists uniquely, and the constant is independent of initial value, so the solution can be extended to the interval $[0, nT]$ for natural number by using the step by step method, and so the proof is complete. □

Hence, from Theorem 4.1 it follows that there exists unique solution $x \in L^2(0,T; W_0^{1,2}(\Omega))$ for every $g \in Z \equiv H_{\frac{1}{2},2} \times L^2(0,T; W_0^{1,2}(\Omega))$ where $H_{\frac{1}{2},2} \equiv (W_0^{1,2}(\Omega), W^{-1,2}(\Omega))_{\frac{1}{2},2}$ (see also [5; Theorem 3.1]). It is also easily seen that

$$L^2(\Omega) = H_{\frac{1}{2},2} = \{\phi \in W^{-1,2}(\Omega) : \int_0^\infty ||A_2 e^{-tA_2}||^2_{W^{-1,2}(\Omega)} dt < \infty\}.$$

Let $x(t; g, f, u)$ be a solution of the equation (4.1) and (4.2) associated with nonlinear term f and control $B_0 u$ at time t. In view of the result of Theorem 4.1 considered as an equation in $W^{-1,2}(\Omega)$, we can define the solution semigroup for the problem (4.1) and (4.2) as follows:

$$S(t)g = (x(t; g, 0, 0), x_t(\cdot; g, 0, 0)) \tag{4.7}$$

where $g = (g^0, g^1) \in Z$, $x(t; g, 0, 0)$ is the solution of (4.1) and (4.2) with $f(t, x) = 0$ and $B_0 = 0$ and $x_t(s; g, 0, 0) = x(t + s; g, 0, 0)$ defined in $[-h, 0]$. It is known that the operator $S(t)$ is a C_0-semigroup on Z. and the infinitesimal generator A of $S(t)$ is characterized by

$$\begin{aligned} D(A) =&\{g = (g^0, g^1) : g^0 \in H,\ g^1 \in L^2(-h, 0; V), \\ &\qquad g^1(0) = g^0,\ A_2 g^0 + A_2 g^1(-h) + \int_{-h}^{0} a(s) A_2 g^1(s) ds \in H\}, \\ Ag =&(A_2 g^0 + A_2 g^1(-h) + \int_{-h}^{0} a(s) A_2 g^1(s) ds, \dot{g}^1). \end{aligned}$$

Note that $a(\cdot)$ need not be Hölder continuous for the above results to hold. It has only to belong to $L^2(-h, 0)$.

For the sake of simplicity, we assume that $S(t)$ is uniformly bounded, that is, there exists a constant $M \geq 1$ such that

$$||S(t)||_Z \leq M. \tag{4.8}$$

as is seen in [5], the equation (4.1) and (4.2) can be transformed into an abstract equation

$$z(t) = Az(t) + F(z(t)) + Bu(t), \tag{4.9}$$

$$z(0) = g, \tag{4.10}$$

where $z(t) = (x(t), x_t(\cdot))$ belongs to the reflexive space Z and $g = (g^0, g^1) \in Z$. The operator A is the infinitesimal generator of C_0-semigroup $S(t)$, $F(z(t)) = (f(t, x(t)), 0)$ and $Bu = (B_0 u, 0)$. The mild solution of initial problem (4.9) and (4.10) is the following form:

$$z(t; g, f, u) = S(t)g + \int_0^t S(t-s)F(z(s))ds + \int_0^t S(t-s)Bu(s)ds.$$

Moreover $\Pi_0 z(t; g, f, u) = x(t; g, f, u)$ where Π_0 denotes the projection of Z onto $L^2(\Omega)$. Since Ω is bounded, the embedding of $W_0^{1,2}(\Omega)$ to $L^2(\Omega)$ is compact, it follows from [1; Theorem 3.4] that the system of generalized eigenspaces of A_2 is complete in $L^2(\Omega)$. Hence just as Theorem 1 and 2 of [4] we can imply the condition (H). Therefore, from Theorems in section 2 we can obtain the following result.

THEOREM 4.2. (1) *The system* (4.1) *and* (4.2) *is approximately controllable in time* T *in* $L^2(\Omega)$.
(2) *The system* (4.1) *and* (4.2) *is approximately controllable in* $L^2(\Omega)$.
(3) *The linear system* (4.1) *and* (4.2) *is approximately controllable in time* T *in* $L^2(\Omega)$.
(4) *The linear system* (4.1) *and* (4.2) *is approximately controllable in* $L^2(\Omega)$.

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