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FUZZY SEMIREGULARIZATION SPACES

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ABSTRACT. We introduce the fuzzy semiregularization space induced by a fuzzy topological space and investigate some properties of fuzzy semiregularization spaces. We give an example of a fuzzy semiregularization space that is not a fuzzy semiregular space.

1. Introduction and preliminaries

A. P. Sostak [13] introduced the fuzzy topology as an extension of Chang's fuzzy topology [3]. It has been developed in many directions [4,5,6,8]. M. N. Mukherjee and B. Ghosh [10] introduced the fuzzy semiregularization space induced by Chang's fuzzy topology.

In this paper, we define the fuzzy semiregularization space induced by a fuzzy topological space in view of A. P. Sostak [13]. We investigate some properties of fuzzy semiregularization spaces. We study the relationships between fuzzy semiregularization spaces and fuzzy semiregular spaces. In [10] the fuzzy semiregularization space induced by Chang's fuzzy topology is a fuzzy semiregular space. But the fuzzy semiregularization space in our sense is not a fuzzy semiregular space. We give an example of it.

Throughout this paper, let X be a nonempty set, $I = [0,1]$ and $I_0 = (0,1]$. All the other notations and the other definitions are standard in a fuzzy set theory.

DEFINITION 1.1 ([13]). A function $\tau : I^X \to I$ is called a *fuzzy topology* on X if it satisfies the following conditions:

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(O1) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$, where $\tilde{0}(x) = 0$ and $\tilde{1}(x) = 1$ for all $x \in X$,
(O2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$,
(O3) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma} \subset I^X$.
The pair (X, τ) is called a *fuzzy topological space* (fts, for short).

DEFINITION 1.2 ([8]). Let $\tilde{0} \notin \Theta$ be a subset of I^X. A function $\beta : \Theta \to I$ is called a *base* for a fuzzy topology on X if it satisfies the following conditions:
(B1) $\beta(\tilde{1}) = 1$,
(B2) $\beta(\mu_1 \wedge \mu_2) \geq \beta(\mu_1) \wedge \beta(\mu_2)$, for any $\mu_1, \mu_2 \in \Theta$.

A base β always *generates* a fuzzy topology on X in the following sense.

THEOREM 1.3. *Let a function $\beta : \Theta \to I$ be a base on X. Define a function $\tau_\beta : I^X \to I$ as follows: for each $\mu \in I^X$,*

$$\tau_\beta(\mu) = \begin{cases} \bigvee(\bigwedge_{j\in J} \beta(\mu_j)) & \text{if } \mu = \bigvee_{j\in J} \mu_j,\ \mu_j \in \Theta, \\ 1 & \text{if } \mu = \tilde{0}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\bigvee$ is taken over all families $\{\mu_j \in \Theta \mid \mu = \bigvee_{j\in J} \mu_j\}$. Then (X, τ_β) is a fts.

Proof. (O1) It is trivial from the definition of τ_β.
(O2) Suppose there exist $\mu, \nu \in I^X$ and $r \in I_0$ such that

$$\tau_\beta(\mu \wedge \nu) < r < \tau_\beta(\mu) \wedge \tau_\beta(\nu).$$

Since $\tau_\beta(\mu) > r$ and $\tau_\beta(\nu) > r$, there exist families $\{\mu_j \in \Theta \mid \mu = \bigvee_{j\in J} \mu_j\}$ and $\{\nu_k \in \Theta \mid \nu = \bigvee_{k\in K} \nu_k\}$ such that

$$\tau_\beta(\mu) \geq \bigwedge_{j\in J} \beta(\mu_j) > r \text{ and } \tau_\beta(\nu) \geq \bigwedge_{k\in K} \beta(\nu_k) > r.$$

Since the unit interval I is completely distributive (ref.[9]), we have

$$\mu \wedge \nu = (\bigvee_{j\in J} \mu_j) \wedge (\bigvee_{k\in K} \nu_k) = \bigvee_{j,k} (\mu_j \wedge \nu_k).$$

Moreover, since $\beta(\mu_j \wedge \nu_k) \geq \beta(\mu_j) \wedge \beta(\nu_k)$, we have

$$\tau_\beta(\mu \wedge \nu) \geq \bigwedge_{j,k}(\beta(\mu_j) \wedge \beta(\nu_k))$$
$$= (\bigwedge_{j \in J} \beta(\mu_j)) \wedge (\bigwedge_{k \in K} \beta(\nu_k)) > r.$$

It is contradiction. Hence $\tau_\beta(\mu_1 \wedge \mu_2) \geq \tau_\beta(\mu_1) \wedge \tau_\beta(\mu_2)$, for all $\mu_1, \mu_2 \in I^X$.

(O3) Suppose there exists a family $\{\mu_j \in \Theta\}_{j \in J}$ and $r \in I_0$ such that

$$\tau_\beta(\bigvee_{j \in J} \mu_j) < r < \bigwedge_{j \in J} \tau_\beta(\mu_j).$$

Since $\tau_\beta(\mu_j) > r$ for each $j \in J$, there exists a family $\{\mu_{jk} \in \Theta \mid \mu_j = \bigvee_{k \in K_j} \mu_{jk}\}$ such that

$$\tau_\beta(\mu_j) \geq \bigwedge_{k \in K_j} \beta(\mu_{jk}) > r.$$

Since $\bigvee_{j \in J} \mu_j = \bigvee_{j \in J}(\bigvee_{k \in K_j}(\mu_{jk}))$, we have

$$\tau_\beta(\bigvee_{j \in J} \mu_j) \geq \bigwedge_{j \in J}(\bigwedge_{k \in K_j} \beta(\mu_{jk})) \geq r.$$

It is a contradiction. Hence $\tau_\beta(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \tau_\beta(\mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset I^X$. □

DEFINITION 1.4. If β is a base on X, then τ_β is called the fuzzy topology *generated* by β. (X, τ_β) is called a fuzzy topological space *generated* by a base β on X.

THEOREM 1.5 ([4]). *Let (X, τ) be a fts. For each $r \in I_0$ and $\lambda \in I^X$, we define a function $C_\tau : I^X \times I_0 \to I^X$ as follows:*

$$C_\tau(\lambda, r) = \bigwedge\{\rho \mid \lambda \leq \rho,\ \tau(\tilde{1} - \rho) \geq r\}.$$

Then it satisfies the following properties: for $\lambda, \mu \in I^X$ and $r, s \in I_0$,
(C1) $C_\tau(\tilde{0}, r) = \tilde{0}$,
(C2) $\lambda \leq C_\tau(\lambda, r)$,
(C3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$,
(C4) $C_\tau(\lambda, r) \leq C_\tau(\lambda, s)$, *if* $r \leq s$,
(C5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

THEOREM 1.6. *Let (X, τ) be a fts. For each $r \in I_0, \lambda \in I^X$, we define a function $I_\tau : I^X \times I_0 \to I^X$ as follows:*

$$I_\tau(\lambda, r) = \bigvee \{\mu \mid \mu \leq \lambda, \tau(\mu) \geq r\}.$$

Then:
(1) $I_\tau(\tilde{1} - \lambda, r) = \tilde{1} - C_\tau(\lambda, r)$.
(2) *For each $\lambda, \mu \in I^X$ and $r, s \in I_0$, we have the followings:*
(I1) $I_\tau(\tilde{1}, r) = \tilde{1}$,
(I2) $I_\tau(\lambda, r) \leq \lambda$,
(I3) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$,
(I4) $I_\tau(\lambda, s) \leq I_\tau(\lambda, r)$, *if* $r \leq s$,
(I5) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.

Proof. (1) For each $\lambda \in I^X$, $r \in I_0$, we have the following:

$$\begin{aligned}\tilde{1} - C_\tau(\lambda, r) &= \tilde{1} - \bigwedge\{\mu \mid \mu \geq \lambda, \tau(\tilde{1} - \mu) \geq r\} \\ &= \bigvee\{\tilde{1} - \mu \mid \mu \geq \lambda, \tau(\tilde{1} - \mu) \geq r\} \\ &= \bigvee\{\tilde{1} - \mu \mid \tilde{1} - \mu \leq \tilde{1} - \lambda, \tau(\tilde{1} - \mu) \geq r\} \\ &= I_\tau(\tilde{1} - \lambda, r).\end{aligned}$$

(2) We easily prove it from Theorem 1.5. □

2. Fuzzy semiregularization spaces

In this section, we study the relationships between fuzzy semiregularization spaces and fuzzy semiregular spaces.

DEFINITION 2.1. Let (X, τ) be a fts. A fuzzy set $\lambda \in I^X$ is said to be *fuzzy regularly open* if there exists $r_0 \in I_0$ such that $\lambda = I_\tau(C_\tau(\lambda, r), r)$ for all $r \leq r_0$. A fuzzy set $\mu \in I^X$ is said to be *fuzzy regularly closed* if there exists $r_1 \in I_0$ such that $\mu = C_\tau(I_\tau(\mu, r), r)$ for all $r \leq r_1$.

LEMMA 2.2. *Let (X, τ) be a fts. Then we have the following statements:*

(1) *A fuzzy set $\lambda \in I^X$ is fuzzy regularly open iff $\tilde{1} - \lambda$ is fuzzy regularly closed.*

(2) *If $\lambda = I_\tau(C_\tau(\lambda, r), r)$, for all $r \leq r_0$, then $\lambda = I_\tau(\lambda, r)$ for all $r \leq r_0$.*

Proof. (1) We easily prove it from the following result: for all $r \leq r_0$, $\lambda = I_\tau(C_\tau(\lambda, r), r)$, we have

$$\begin{aligned}\tilde{1} - \lambda &= \tilde{1} - I_\tau(C_\tau(\lambda, r), r)\\ &= C_\tau(\tilde{1} - C_\tau(\lambda, r), r) \quad \text{(by Theorem 1.6(1))}\\ &= C_\tau(I_\tau(\tilde{1} - \lambda, r), r).\end{aligned}$$

(2) For all $r \leq r_0$, we have $\lambda = I_\tau(C_\tau(\lambda, r), r)$. Hence $I_\tau(\lambda, r) = I_\tau(I_\tau(C_\tau(\lambda, r), r), r)$ and

$$\begin{aligned}I_\tau(\lambda, r) &= I_\tau(C_\tau(\lambda, r), r)\\ &= \lambda\end{aligned}$$

from (I5) of Theorem 1.6(2). □

EXAMPLE 2.3. Let $X = \{a, b\}$ be a set. Let $\mu, \rho \in I^X$ as follows:

$$\mu(a) = 0.3,\ \mu(b) = 0.4, \quad \rho(a) = 0.6,\ \rho(b) = 0.2.$$

We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1},\\ \frac{1}{2}, & \text{if } \lambda = \mu,\\ \frac{2}{3}, & \text{if } \lambda = \rho,\\ \frac{2}{3}, & \text{if } \lambda = \mu \wedge \rho,\\ \frac{1}{2}, & \text{if } \lambda = \mu \vee \rho,\\ 0, & \text{otherwise.}\end{cases}$$

From Theorem 1.5 and Theorem 1.6, we obtain the following:

$$\begin{aligned}
\tilde{1} &= I_\tau(C_\tau(\tilde{1}, r), r), && \forall r \in I_0, \\
\mu &= I_\tau(C_\tau(\mu, r), r), && 0 < r \leq \frac{1}{2}, \\
\mu \vee \rho &= I_\tau(C_\tau(\rho, r), r), && 0 < r \leq \frac{1}{2}, \\
\rho &= I_\tau(C_\tau(\rho, r), r), && \frac{1}{2} < r \leq \frac{2}{3}, \\
\mu &= I_\tau(C_\tau(\mu \wedge \rho, r), r), && 0 < r \leq \frac{1}{2}, \\
\mu \wedge \rho &= I_\tau(C_\tau(\mu \wedge \rho, r), r), && \frac{1}{2} < r \leq \frac{2}{3}, \\
\mu \vee \rho &= I_\tau(C_\tau(\mu \vee \rho, r), r), && 0 < r \leq \frac{1}{2}.
\end{aligned}$$

Hence μ and $\mu \vee \rho$ are fuzzy regularly open. But ρ and $\mu \wedge \rho$ are not fuzzy regularly open. We have $\rho = I_\tau(\rho, r)$, $0 < r \leq \frac{2}{3}$ but $\rho \neq I_\tau(C_\tau(\rho, r), r), 0 < r \leq \frac{2}{3}$. Hence the converse of Lemma 2.2(2) is not true.

REMARK 2.4. We define that $\mu \in I^X$ is fuzzy regularly open iff $\lambda = I_\tau(C_\tau(\lambda, r), r)$. Let $\tau_r = \{\mu \mid \mu \text{ is fuzzy regularly open set}, \tau(\mu) \geq r\}$. In Example 2.3, since $\rho \in \tau_{\frac{2}{3}}$ but $\rho \notin \tau_{\frac{1}{2}}$, then $\{\tau_r \mid r \in I_0\}$ is not a descending family (ref.[6]). In this case, we cannot naturally define the fuzzy topology generated by fuzzy regularly open sets.

In the following theorem, we construct the fuzzy topology generated by fuzzy regularly open sets.

THEOREM 2.5. *Let (X, τ) be a fts and $\tilde{0} \notin \Theta_\tau$ a family of all fuzzy regularly open sets. Define a function $\beta_\tau : \Theta_\tau \to I$ by*

$$\beta_\tau(\lambda) = \bigvee \{r \in I_0 \mid \lambda = I_\tau(C_\tau(\lambda, r), r)\}.$$

Then β_τ is a base on X such that $\tau_{\beta_\tau} \leq \tau$.

Proof. First, we will show that β_τ is a base on X.

(B1) For all $r \in I_0$, we have $\tilde{1} = I_\tau(C_\tau(\tilde{1}, r), r)$ because $C_\tau(\tilde{1}, r) = \tilde{1}$ from Theorem 1.5(C2). Hence $\beta_\tau(\tilde{1}) = 1$.

(B2) Suppose there exist $\lambda_1, \lambda_2 \in \Theta_\tau$ such that

$$\beta_\tau(\lambda_1 \wedge \lambda_2) < \beta_\tau(\lambda_1) \wedge \beta_\tau(\lambda_2).$$

From the definition of β_τ, there exist $r_i \in I_0$ for $i \in \{1, 2\}$ with for each $0 < r \leq r_i$,

$$\lambda_i = I_\tau(C_\tau(\lambda_i, r), r)$$

such that

$$\beta_\tau(\lambda_1 \wedge \lambda_2) < r_1 \wedge r_2 \leq \beta_\tau(\lambda_1) \wedge \beta_\tau(\lambda_2).$$

Put $r_0 = r_1 \wedge r_2$. Since $I_\tau(\lambda_i, r) = \lambda_i$ for all $0 < r \leq r_i$ from Lemma 2.2(2), we have for each $0 < r \leq r_0$,

$$(A) \qquad \begin{aligned} I_\tau(C_\tau(\lambda_1 \wedge \lambda_2, r), r) &\geq I_\tau(\lambda_1 \wedge \lambda_2, r) \\ &= I_\tau(\lambda_1, r) \wedge I_\tau(\lambda_2, r) \\ &= \lambda_1 \wedge \lambda_2. \end{aligned}$$

On the other hand, since $\lambda_i = I_\tau(C_\tau(\lambda_i, r), r)$, for each $0 < r \leq r_0$, we have

$$(B) \qquad \begin{aligned} I_\tau(C_\tau(\lambda_1 \wedge \lambda_2, r), r) &\leq I_\tau(C_\tau(\lambda_1, r) \wedge C_\tau(\lambda_2, r), r) \\ &= I_\tau(C_\tau(\lambda_1, r), r) \wedge I_\tau(C_\tau(\lambda_2, r), r) \\ &= \lambda_1 \wedge \lambda_2. \end{aligned}$$

From (A) and (B), we have for each $0 < r \leq r_0$,

$$\lambda_1 \wedge \lambda_2 = I_\tau(C_\tau(\lambda_1 \wedge \lambda_2, r), r).$$

Thus $\beta_\tau(\lambda_1 \wedge \lambda_2) \geq r_0$. It is a contradiction. Hence $\beta_\tau(\mu_1 \wedge \mu_2) \geq \beta_\tau(\mu_1) \wedge \beta_\tau(\mu_2)$, for each $\mu_1, \mu_2 \in \Theta_\tau$.

Finally, we will show that $\tau_{\beta_\tau} \leq \tau$. Suppose there exist $\lambda \in I^X$ and $r_1 \in I_0$ such that

$$\tau_{\beta_\tau}(\lambda) > r_1 > \tau(\lambda).$$

From Theorem 1.3, there exists a family $\{\lambda_i \in \Theta_\tau \mid \lambda = \bigvee_{i\in\Gamma} \lambda_i\}$ such that

$$\tau_{\beta_\tau}(\lambda) \geq \bigwedge_{i\in\Gamma} \beta_\tau(\lambda_i) > r_1 > \tau(\lambda).$$

For each $i \in \Gamma$, since $\beta_\tau(\lambda_i) > r_1$, there exists $r_i \in I_0$ with for each $0 < r \leq r_i$,

$$\lambda_i = I_\tau(C_\tau(\lambda_i, r), r)$$

such that

$$\beta_\tau(\lambda_i) \geq r_i > r_1.$$

On the other hand, for $i \in \Gamma$, since $I_\tau(\lambda_i, r_i) = \lambda_i$ from Lemma 2.2(2), we have $\tau(\lambda_i) \geq r_i$. Thus

$$\tau(\lambda) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i) \geq \bigwedge_{i\in\Gamma} r_i \geq r_1.$$

It is a contradiction. Hence $\tau_{\beta_\tau} \leq \tau$. □

EXAMPLE 2.6. In Example 2.3, we have $\Theta_\tau = \{\tilde{1}, \mu, \mu \vee \rho\}$ and τ_{β_τ} as follows:

$$\tau_{\beta_\tau}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ \frac{1}{2}, & \text{if } \lambda = \mu \vee \rho, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we have $\tau_{\beta_\tau} \leq \tau$.

NOTATION. We simply write τ_s instead of τ_{β_τ}.

DEFINITION 2.7. A fts (X, τ_s) is said to be *the fuzzy semiregularization space* (fsrs, for short) of (X, τ). A fts (X, τ) is said to be *fuzzy semiregular* if $\tau = \tau_s$.

LEMMA 2.8. *Let (X, τ_s) be the fsrs of a fts (X, τ). Then:*

(1) $I_\tau(C_\tau(\lambda, r), r) = I_\tau(C_\tau(I_\tau(C_\tau(\lambda, r), r), r), r)$, *for all* $\lambda \in I^X$ *and* $r \in I_0$.

(2) *If* $\rho = I_\tau(C_\tau(\lambda, r), r)$ *for* $0 < r \leq r_0$, *then* $\tau_s(\rho) \geq r_0$.

(3) *If* $\mu = C_\tau(I_\tau(\lambda, r), r)$ *for* $0 < r \leq r_0$, *then* $\mu = C_{\tau_s}(I_\tau(\lambda, r), r)$ *for* $0 < r \leq r_0$.

(4) *If* $\rho = I_\tau(C_\tau(\lambda, r), r)$ *and* $C_{\tau_s}(\lambda, r) = C_\tau(\lambda, r)$ *for* $0 < r \leq r_0$, *then* $\rho = I_{\tau_s}(C_{\tau_s}(\lambda, r), r)$ *for* $0 < r \leq r_0$.

Proof. (1) Since $I_\tau(C_\tau(\lambda, r), r) \leq C_\tau(I_\tau(C_\tau(\lambda, r), r), r)$ from (C2) of Theorem 1.5, we have

$$(D) \qquad I_\tau(C_\tau(\lambda, r), r) \leq I_\tau(C_\tau(I_\tau(C_\tau(\lambda, r), r), r), r).$$

Conversely, since $I_\tau(C_\tau(\lambda, r), r) \leq C_\tau(\lambda, r)$, we have

$$(E) \qquad \begin{aligned} &C_\tau(I_\tau(C_\tau(\lambda, r), r), r) \leq C_\tau(\lambda, r) \\ \Rightarrow &I_\tau(C_\tau(I_\tau(C_\tau(\lambda, r), r), r), r) \leq I_\tau(C_\tau(\lambda, r), r). \end{aligned}$$

Thus, by (D) and (E), we have

$$I_\tau(C_\tau(\lambda, r), r) = I_\tau(C_\tau(I_\tau(C_\tau(\lambda, r), r), r), r).$$

(2) From (1), put $\rho = I_\tau(C_\tau(\lambda, r), r)$ for $0 < r \leq r_0$. Then, for $0 < r \leq r_0$,

$$\rho = I_\tau(C_\tau(\rho, r), r).$$

Hence ρ is fuzzy regularly open. Thus $\tau_s(\rho) \geq r_0$.

(3) Since, by Theorem 1.6(1), for $0 < r \leq r_0$,

$$\begin{aligned} \tilde{1} - \mu &= \tilde{1} - C_\tau(I_\tau(\lambda, r), r) \\ &= I_\tau(\tilde{1} - I_\tau(\lambda, r), r) \\ &= I_\tau(C_\tau(\tilde{1} - \lambda, r), r), \end{aligned}$$

by (2), $\tau_s(\tilde{1} - \mu) \geq r_0$. It implies that for $0 < r \leq r_0$,

$$\begin{aligned} C_{\tau_s}(I_\tau(\lambda, r), r) &= \bigwedge\{\rho \mid I_\tau(\lambda, r) \leq \rho, \ \ \tau_s(\tilde{1} - \rho) \geq r\} \\ &\leq \mu. \end{aligned}$$

On the other hand, we have for $0 < r \leq r_0$,

$$\begin{aligned} C_{\tau_s}(I_\tau(\lambda,r),r) &= \bigwedge\{\rho \mid I_\tau(\lambda,r) \leq \rho,\ \ \tau_s(\tilde{1}-\rho) \geq r\} \\ &\geq \bigwedge\{\rho \mid I_\tau(\lambda,r) \leq \rho,\ \ \tau(\tilde{1}-\rho) \geq r\} \quad (\tau_s \leq \tau) \\ &= C_\tau(I_\tau(\lambda,r),r) = \mu. \end{aligned}$$

Hence $\mu = C_{\tau_s}(I_\tau(\lambda,r),r)$, for $0 < r \leq r_0$.

(4) Let $\rho = I_\tau(C_\tau(\lambda,r),r)$ for $0 < r \leq r_0$. Then for $0 < r \leq r_0$,

$$\begin{aligned} \tilde{1}-\rho &= \tilde{1} - I_\tau(C_\tau(\lambda,r),r) \\ &= C_\tau(\tilde{1} - C_\tau(\lambda,r),r) \\ &= C_\tau(I_\tau(\tilde{1}-\lambda,r),r) \\ &= C_{\tau_s}(I_\tau(\tilde{1}-\lambda,r),r). \quad \text{(by (3))} \end{aligned}$$

It implies, for $0 < r \leq r_0$,

$$\begin{aligned} \rho &= \tilde{1} - C_{\tau_s}(I_\tau(\tilde{1}-\lambda,r),r) \\ &= I_{\tau_s}(\tilde{1} - I_\tau(\tilde{1}-\lambda,r),r) \\ &= I_{\tau_s}(C_\tau(\lambda,r),r) \\ &= I_{\tau_s}(C_{\tau_s}(\lambda,r),r) \end{aligned}$$

since $C_{\tau_s}(\lambda,r) = C_\tau(\lambda,r)$. □

THEOREM 2.9. *Let (X,τ_s) be the fsrs of a fts (X,τ) and $\beta_\tau : \Theta_\tau \to I$ be a base of the fsrs (X,τ_s). If $C_{\tau_s}(\lambda_i,r) = C_\tau(\lambda_i,r)$ for $0 < r \leq r_i$ such that $\beta_\tau(\lambda_i) = r_i$ for each $\lambda_i \in \Theta_\tau$, Then (X,τ_s) is fuzzy semiregular.*

Proof. We only show that $\tau_s \leq (\tau_s)_s$ because $\tau_s \geq (\tau_s)_s$ from Theorem 2.5. Suppose there exist $\lambda \in I^X$ and $r_1 \in I_0$ such that

$$\tau_s(\lambda) > r_1 > (\tau_s)_s(\lambda).$$

From Theorem 1.3, there exists a family $\{\lambda_i \in \Theta_\tau \mid \lambda = \bigvee_{i\in\Gamma}\lambda_i\}$ such that

$$\tau_s(\lambda) \geq \bigwedge_{i\in\Gamma} \beta_\tau(\lambda_i) > r_1 > (\tau_s)_s(\lambda).$$

For each $i \in \Gamma$, since $\beta_\tau(\lambda_i) > r_1$, there exists $r_i \in I_0$ with for each $0 < r \leq r_i$,

$$\lambda_i = I_\tau(C_\tau(\lambda_i, r), r)$$

such that

$$\beta_\tau(\lambda_i) \geq r_i > r_1.$$

Since $C_{\tau_s}(\lambda_i, r) = C_\tau(\lambda_i, r)$ for $0 < r \leq r_i$, we have

$$\begin{aligned} \lambda_i &= I_\tau(C_\tau(\lambda_i, r), r) \\ &= I_{\tau_s}(C_{\tau_s}(\lambda_i, r), r). \text{ (by Lemma 2.8(4))} \end{aligned}$$

It implies $\lambda_i \in \Theta_{\tau_s}$ with $\beta_{\tau_s}(\lambda_i) \geq r_i$. Thus

$$(\tau_s)_s(\lambda) \geq \bigwedge_{i \in \Gamma} \beta_{\tau_s}(\lambda_i) \geq \bigwedge_{i \in \Gamma} r_i \geq r_1.$$

It is a contradiction. Hence $\tau_s \leq (\tau_s)_s$. □

EXAMPLE 2.10. From Example 2.3 and Example 2.6, we can obtain the fsrs (X, τ_s) of (X, τ) as follows:

$$\tau_s(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ \frac{1}{2}, & \text{if } \lambda = \mu \vee \rho, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \tilde{1} &= I_{\tau_s}(C_{\tau_s}(\tilde{1}, r), r), && \forall r \in I_0, \\ \mu &= I_{\tau_s}(C_{\tau_s}(\mu, r), r), && 0 < r \leq \frac{1}{2}, \\ \mu \vee \rho &= I_{\tau_s}(C_{\tau_s}(\mu \vee \rho, r), r), && 0 < r \leq \frac{1}{2}. \end{aligned}$$

We obtain the fsrs $(X, (\tau_s)_s)$ of (X, τ_s) as follows:

$$(\tau_s)_s(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ \frac{1}{2}, & \text{if } \lambda = \mu \vee \rho, \\ 0, & \text{otherwise.} \end{cases}$$

Hence (X, τ_s) is fuzzy semiregular because $(\tau_s)_s = \tau_s$. It satisfies the condition of Theorem 2.9 from the following:

$$\tilde{1} - (\mu \vee \rho) = C_{\tau_s}(\mu, r) = C_\tau(\mu, r), \qquad 0 < r \leq \frac{1}{2},$$
$$\tilde{1} - \mu = C_{\tau_s}(\mu \vee \rho, r) = C_\tau(\mu \vee \rho, r), 0 < r \leq \frac{1}{2}.$$

Let X, μ and ρ be defined as Example 2.3. We define another fuzzy topology $\tau^* : I^X \to I$ as follows:

$$\tau^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ \frac{2}{3}, & \text{if } \lambda = \rho, \\ \frac{3}{4}, & \text{if } \lambda = \mu \wedge \rho, \\ \frac{3}{5}, & \text{if } \lambda = \mu \vee \rho, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain

$$\tilde{1} = I_{\tau^*}(C_{\tau^*}(\tilde{1}, r), r), \qquad \forall r \in I_0,$$
$$\mu = I_{\tau^*}(C_{\tau^*}(\mu, r), r), \qquad 0 < r \leq \frac{1}{2},$$
$$\rho = I_{\tau^*}(C_{\tau^*}(\rho, r), r), \qquad \frac{3}{5} < r \leq \frac{2}{3},$$
$$\mu \wedge \rho = I_{\tau^*}(C_{\tau^*}(\mu \wedge \rho, r), r), \frac{1}{2} < r \leq \frac{3}{4},$$
$$\mu \vee \rho = I_{\tau^*}(C_{\tau^*}(\mu \vee \rho, r), r), 0 < r \leq \frac{3}{5}.$$

Thus, we have

$$\tau_s^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ \frac{3}{5}, & \text{if } \lambda = \mu \vee \rho, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, since

$$\begin{aligned} \tilde{1} &= I_{\tau_s^*}(C_{\tau_s}(\tilde{1}, r), r), \qquad \forall r \in I_0, \\ \mu &= I_{\tau_s^*}(C_{\tau_s^*}(\mu, r), r), \qquad 0 < r \leq \frac{1}{2}, \\ \mu \vee \rho &= I_{\tau_s^*}(C_{\tau_s^*}(\mu \vee \rho, r), r), \; 0 < \rho \leq \frac{1}{2}, \end{aligned}$$

then

$$(\tau_s^*)_s(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ \frac{1}{2}, & \text{if } \lambda = \mu \vee \rho, \\ 0, & \text{otherwise.} \end{cases}$$

Hence (X, τ_s^*) is not a fuzzy semiregular space because $(\tau_s^*)_s \neq \tau_s^*$. It does not satisfy the condition of Theorem 2.9 from the following:

$$\begin{aligned} \tilde{1} - \mu &= C_{\tau_s^*}(\mu \vee \rho, r) = C_{\tau^*}(\mu \vee \rho, r), \qquad 0 < r \leq \frac{1}{2}, \\ \tilde{1} - (\mu \wedge \rho) &= C_{\tau^*}(\mu \vee \rho, r) \neq C_{\tau_s^*}(\mu \vee \rho, r) = \tilde{1}, \; \frac{1}{2} < r \leq \frac{3}{5}. \end{aligned}$$

REMARK 2.11. In Definition 1.1, we take $\{0, 1\}$ instead of the range I of a function τ. We regard it as Chang's fuzzy topology. In our sense, τ_s is fuzzy semiregular (ref.[10]). The fsrs in our sense is an extension of it in [10].

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