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ON DOUBLY STOCHASTIC k-POTENT MATRICES AND REGULAR MATRICES

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ABSTRACT. In this paper, we determine the structure of k-potent elements and regular elements of the semigroup Ω_n of doubly stochastic matrices of order n. In connection with this, we find the structure of the matrices X satisfying the equation $AXA = A$. From these, we determine a condition of a doubly stochastic matrix A whose Moore-Penrose generalized inverse is also a doubly stochastic matrix.

0. Introduction

Let $\mathbf{R}^n$ denote the set of all n-dimensional real column vectors, and let $\mathbf{e}$ denote the vector in $\mathbf{R}^n$ all of whose entries are 1. If all of the entries of a real matrix A are nonnegative, A is called *nonnegative matrix* and is denoted by $A \geq O$. An $n \times n$ nonnegative matrix D is said to be *doubly stochastic* if $D\mathbf{e} = \mathbf{e}, \mathbf{e}^T D = \mathbf{e}^T$. Let J_n denote the $n \times n$ doubly stochastic matrix all of whose entries are $1/n$. As usual, we denote the set of all $n \times n$ doubly stochastic matrices by Ω_n. The set Ω_n has very rich and interesting combinatorial and geometric properties. For instance, there are a lot of results for permanent of elements in Ω_n [6,7,8], and it is very well known that Ω_n is a convex polyhedron whose vertices are all permutation matrices [See (1) chapter 2]. In addition, Ω_n is known to be a compact semigroup under the ordinary matrix multiplication with respect to the natural topology [4], and is very closely related to the theory of majorization [5].

In earlier results, it was proved that there is a finite number of maximal subgroups of the semigroup Ω_n [1], and all idempotent elements are

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obtained in [2]. In 1983, J. S. Montague and R. J. Plemmons gave several equivalence relations for regular elements in Ω_n, and characterized the Green's relation on the semigroup Ω_n [9]. S. Schwarz [11] and H. K. Farahat [3] have shown that the maximal subgroups of Ω_n are direct sum of symmetric groups.

In this paper, we explicitly determine the structure of regular elements and k-potent elements of the semigroup Ω_n by using Perron-Frobenius Theorem.

1. Preliminaries

An $n \times n$ matrix A is said to be *reducible* if there exists a permutation matrix P such that

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are nonvacuous square matrices, otherwise it is called *irreducible* .

For each square matrix B, there exists a permutation matrix Q such that

$$Q^T BQ = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ O & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_{kk} \end{bmatrix}$$

where B_{ii} is an irreducible square matrix for each $i = 1, \cdots, k$. The matrices B_{ii} are irreducible components of B. If a doubly stochastic matrix D is reducible, then it is permutation similar to a direct sum of irreducible doubly stochastic matrices.

The following theorem about the spectral radius of nonnegative irreducible matrices is very well known. For a real square matrix A, let $\rho(A)$ denote the spectral radius of A.

THEOREM 1 (Perron-Frobenius). *If A is a square nonnegative and irreducible matrix, then*

(a) *$\rho(A)$ is an eigenvalue of A,*

(b) *There is a positive eigenvector corresponding to the eigenvalue $\rho(A)$,*

(c) *Algebraic multiplicity of* $\rho(A)$ *as an eigenvalue of* A *is* 1.

By Theorem 1 and Geršgorin Theorem, 1 is the simple and maximal eigenvalue of all irreducible doubly stochastic matrix. Since a reducible doubly stochastic matrix is permutation similar to a direct sum of doubly stochastic matrices, we get the following.

LEMMA 2. *Algebraic multiplicity of the eigenvalue* 1 *of a doubly stochastic matrix is the number of its irreducible components.*

2. k-potent elements and regular elements of Ω_n

An element E in Ω_n is said to be *idempotent* if $E^2 = E$, and it is said to be *k-potent* if $E^k = E$ but $E^m \neq E$ for $m = 2, \cdots, k-1$. It is very well known that an idempotent matrix is diagonalizable, and each of its eigenvalues is either 1 or 0. Therefore the rank of an idempotent matrix is the same as its trace. Note J_n is the only doubly stochastic matrix of rank 1. From these facts and from Lemma 2, we obtain the following. The following theorem due to J. L. Doob [2].

THEOREM 3 (Doob). *An* $n \times n$ *doubly stochastic matrix* A *is idempotent if and only if there exists a permutation matrix* P *such that*

$$P^T AP = J_{k_1} \oplus J_{k_2} \oplus \cdots \oplus J_{k_r}$$

where $k_1 + k_2 + \cdots + k_r = n$.

For an $r \times r$ permutation matrix P and square matrices $A_1, A_2, \cdots, A_r$ we denote by $\otimes(P; A_1, \cdots, A_r)$ the matrix obtained from P replacing the 1 in the ith row by A_i for each $i = 1, 2, \cdots, n$, and each of the zero entries by a zero matrix of suitable size. For example, if

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

then

$$\otimes(P; A_1, A_2, A_3) = \begin{bmatrix} O & A_1 & O \\ O & O & A_2 \\ A_3 & O & O \end{bmatrix}.$$

THEOREM 4. *An $n \times n$ doubly stochastic matrix A is k-potent if and only if there exist permutation matrices P, Q such that*

$$P^T AP = \otimes(Q; J_{k_1}, J_{k_2}, \cdots, J_{k_r})$$

where $k_1 + k_2 + \cdots + k_r = n$ and Q is k-potent in which all the irreducible components of Q are of the same order.

Proof. Let $k \geq 3$ be an integer such that $A^k = A$. Then $A^{k-1}A = A, AA^{k-1} = A$. This says that all column vectors (row vectors) of A are right (left, respectively) eigenvectors of A^{k-1} corresponding to eigenvalue 1. If A^{k-1} is irreducible, then $\mathbf{e}$ is the only eigenvector of A^{k-1} corresponding to the eigenvalue 1. So, A must be J_n, an idempotent matrix, and we are done in this case.

Suppose that, for some permutation matrix P, $P^T A^{k-1} P$ is of the form

$$P^T A^{k-1} P = \begin{bmatrix} B_{11} & O & \cdots & O \\ O & B_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_{rr} \end{bmatrix}$$

where each B_{ii} is an irreducible component of A^{k-1} for $i = 1, \cdots, r$. Let $P^T AP$ be partitioned as

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix},$$

where A_{ii} is of the same size as B_{ii} for each $i = 1, 2, \cdots, r$.

Since all column vectors and row vectors of A are right and left eigenvectors of A^{k-1} corresponding to eigenvalue 1, $B_{ii}A_{ij} = A_{ij}$ and $A_{ij}B_{jj} = A_{ij}$ for all $i, j = 1, 2, \cdots, r$. From the irreducibility of B_{ii}, all the entries of each of the blocks A_{ij} are the same for $i, j = 1, 2, \cdots, r$.

Now we partition $P^T A^{k-2} P$ as the same way of $P^T AP$,

$$P^T A^{k-2} P = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ C_{21} & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rr} \end{bmatrix}$$

where each C_{ii} is of the same size as A_{ii} for each $i = 1, 2, \cdots, r$.

Since $AA^{k-2} = A^{k-2}A = A^{k-1}$, we have

$$P^T AA^{k-2} P = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ C_{21} & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rr} \end{bmatrix}$$
$$= \begin{bmatrix} B_{11} & O & \cdots & O \\ O & B_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_{rr} \end{bmatrix} = P^T A^{k-1} P.$$

Since each A_{ij} is either a positive matrix or O matrix and since C_{ij} is a nonnegative matrix and $\sum_{l=1}^{r} A_{il} C_{lj} = O$ for $i \neq j$, we get that $A_{il} C_{lj} = O$ if $i \neq j$.

Suppose that there are two nonzero blocks in the same block row of the above partition $P^T AP$, say A_{11} and A_{12} are nonzero blocks, then all of C_{ij} are O for $i = 1, 2,\ j = 2, \cdots, r$, i.e.

$$P^T A^{k-2} P = \begin{bmatrix} C_{11} & O & O & \cdots & O \\ C_{21} & O & O & \cdots & O \\ C_{31} & C_{32} & C_{33} & \cdots & C_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & C_{r3} & \cdots & C_{rr} \end{bmatrix}$$

Contradicting to that A^{k-2} is doubly stochastic. Therefore each row of $P^T AP$ has exactly one nonnegative block entry, telling us that each nonzero block is a doubly stochastic matrix, so that it is a square matrix. Let A_i denote the nonzero block in the ith row of $P^T AP$. Then $P^T AP$ can be expressed as $\otimes(Q; A_1, A_2, \cdots, A_r)$ for some permutation matrix Q.

Now we determine the size of the each nonzero block. First, we can easily observe that A_i and A_j have the same size if A_i is placed in the position (i,j) of Q. There is a well known fact that a matrix $A=(a_{ij})_{n\times n}$ is irreducible if and only if for every pair of distinct integers p,q with $1\leq p,q\leq n$ there is a sequence of distinct integers $k_1,k_2,\cdots,k_{m-1},k_m=q$ such that all of the matrix entries $a_{pk_1},a_{k_1k_2},\cdots,a_{k_{m-1}q}$ are nonzero.

Thus nonzero blocks in $\otimes(Q;A_1,\cdots,A_r)$ which came from 1's in each irreducible component of Q have the same size.

The converse is obvious. □

In the semigroup Ω_n, an elements $E\in\Omega_n$ is called a *regular element* if $EXE=E$ for some $X\in\Omega_n$. If $XEX=X$ in addition, then E and X are said to be *semi-inverses* each other. In this case, EX and XE are idempotent elements of Ω_n.

In 1973, Montague and Plemmons proved that there are only a finite number of regular elements in Ω_n. What are the regular elements of Ω_n? In a survey paper ([7], see p 253) of H. Minc, it is noted that Sinkhorn [unpublished] showed that A is regular if and only if A is permutation equivalent to $J_{n_1}\oplus J_{n_2}\oplus\cdots\oplus J_{n_r}$ for some positive integers $n_1,n_2,\cdots,n_r$. In the following we give a proof of this fact by an argument similar to that used in the proof of Theorem 4.

If $E\in\Omega_n$ is a regular element, then all column vectors of E are eigenvectors of certain doubly stochastic matrix corresponding to the eigenvalue 1.

Now, we are ready to prove the following theorem.

THEOREM 5. *An $n\times n$ matrix A in Ω_n is regular if and only if there exist permutation matrices P,Q such that*

$$PAQ=J_{k_1}\oplus J_{k_2}\oplus\cdots\oplus J_{k_r}$$

where each k_i's are positive integers with $k_1+k_2+\cdots+k_r=n$.

Proof. Suppose $AXA=A$ where $A,X\in\Omega_n$. Then, by Theorem 3, there exist permutation matrices P_1,P_2 such that

$$P_1{}^TAXP_1=J_{k_1}\oplus\cdots\oplus J_{k_r}, \tag{1}$$

$$P_2{}^TXAP_2=J_{l_1}\oplus\cdots\oplus J_{l_m}. \tag{2}$$

Since AX and XA have the same eigenvalues counting multiplicities, we have $r=m$.

Let $P_1^T A P_2$ be partitioned as

$$P_1^T A P_2 = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix}$$

where each of A_{ij}'s is of size $k_i \times l_j$ for $i, j = 1, 2, \cdots, r$. Since all column vectors (row vectors) of A is right (left, respectively) eigenvectors of AX (XA, respectively) corresponding to the eigenvalue 1, each A_{ij} is either a positive matrix or O. Let $P_2^T X P_1$ be partitioned as that of $(P_1^T A P_2)^T$. Then (1) and (2) can be expressed as

$$\begin{aligned} P_1{}^T A P_2 P_2^T X P_1 &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ C_{21} & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rr} \end{bmatrix} \\ &= \begin{bmatrix} J_{k_1} & O & \cdots & O \\ O & J_{k_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & J_{k_r} \end{bmatrix}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} P_2{}^T X P_1 P_1{}^T A P_2 &= \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ C_{21} & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rr} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix} \\ &= \begin{bmatrix} J_{k_1} & O & \cdots & O \\ O & J_{k_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & J_{k_r} \end{bmatrix}. \end{aligned} \tag{4}$$

Since the block multiplication is well defined, and since all entries of each A_{ij} are the same, we see that each of the A_{ij}'s is either positive or O. Now suppose that there are two nonzero blocks in a same block rows of $P_1^T A P_2$, say $A_{11} \neq O$ and $A_{12} \neq O$, then $P_2{}^T X P_1$ is of the form

$$(5) \qquad P_2{}^T X P_1 = \begin{bmatrix} C_{11} & O & O & \cdots & O \\ C_{21} & O & O & \cdots & O \\ C_{31} & C_{32} & C_{33} & \cdots & C_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & C_{r3} & \cdots & C_{rr} \end{bmatrix}.$$

From (4) and (5), $C_{21}A_{11} = O$, yielding $C_{21} = O$, a contradiction. Thus each block row of $P_1^T AP_2$ has exactly 1 nonzero block. Similarly, there is only one nonzero block in block column of $P_1^T AP_2$. This implies our assertion.

Converse is obvious. □

From the fact that $J_n A = AJ_n = J_n$ and from Theorem 5, we have the following.

COROLLARY 1. *Let $A \in \Omega_n$. If the equation $AXA = A$ is solvable in Ω_n, then*

(1) $PAQ = J_{k_1} \oplus \cdots \oplus J_{k_r}$ for some permutation matrices P and Q

(2) $Q^T X P^T = B_1 \oplus \cdots \oplus B_r$ where $B_i \in \Omega_{k_i}$ for $i = 1, \cdots, r$.

For an $m \times n$ real matrix Y, let Y^+ denote the *Moore-Penrose generalized inverse* of Y , that is , the unique $n \times m$ matrix satisfying $YY^+Y = Y$, $Y^+YY^+ = Y^+$, $(YY^+)^T = YY^T$ and $(Y^+Y)^T = Y^+Y$.

If Y is a square nonsingular matrix, then clearly $Y^+ = Y^{-1}$. Furthermore, if all column vectors of Y are linearly independent, then $Y^+ = (Y^TY)^{-1}Y^T$.

The following corollary directly comes from previous results.

COROLLARY 2. *Let $A \in \Omega_n$. Then the following statements are equivalent.*

(1) A is regular.

(2) $PAQ = J_{k_1} \oplus \cdots \oplus J_{k_r}$ for some permutation matrices P and Q

(3) A^T is the unique semi-inverse of A.

(4) A^+ is nonnegative.

(5) $A^+ = A^T$.

The equivalence of (1), (3), (4), (5) in Corollary 7 is proved in [9].

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