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THE REDUCED MINIMUM MODULUS OF 2×2 UPPER TRIANGULAR OPERATOR MATRICES

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ABSTRACT. When $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are given we denote by M_C an operator acting on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$ of the form

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. In this note we consider the reduced minimum modulus of M_C. The main result is as follows. If A has dense range and B is one-one then $\frac{\gamma(A)\gamma(B)}{\max\{\gamma(A),\gamma(B)\}+\|C\|} \le \gamma(M_C) \le \min\{\gamma(A), \gamma(B)\}$, where $\gamma(\cdot)$ denotes the reduced minimum modulus.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ denote the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, and abbreviate $\mathcal{L}(\mathcal{H}, \mathcal{H})$ to $\mathcal{L}(\mathcal{H})$. If $A \in \mathcal{L}(\mathcal{H})$ write $\sigma(A)$ for the ordinary spectrum of A. If $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ write $N(A)$ for the null space of A; $R(A)$ for the range of A. Recall ([**5**]) that an operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is said to be *bounded below* if there exists $k > 0$ for which $||x|| \le k\,||Ax||$ for each $x \in \mathcal{H}$.

When $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are given we denote by M_C an operator acting on $\mathcal{H} \oplus \mathcal{K}$ of the form

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \tag{0.1}$$

where $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. It is familiar that if an operator T has an invariant subspace then T can be written as (0.1).

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If $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ then the *reduced minimum modulus* of T is defined by (cf. [1])

$$\gamma(T) = \begin{cases} \inf\{||Tx|| : \text{ dist}\big(x, N(T)\big) = 1\} & \text{if } T \neq 0 \\ 0 & \text{if } T = 0. \end{cases}$$

Thus $\gamma(T) > 0$ if and only if T has closed nonzero range (cf. [1], [3]). If $T \in \mathcal{L}(\mathcal{H})$ is a non-zero operator then we can see ([1]) that $\gamma(T) = \inf\big(\sigma(|T|) \setminus \{0\}\big)$, where $|T|$ denotes $(T^*T)^{\frac{1}{2}}$. Thus we have that $\gamma(T) = \gamma(T^*)$. If T is bounded below then $||x|| \leq \frac{1}{\gamma(T)} ||Tx||$ for each $x \in \mathcal{H}$. It is known (cf. [1], [7]) that if T is invertible then $\gamma(T) = \frac{1}{||T^{-1}||}$.

In this note, we consider the reduced minimum modulus of M_C.

We begin with:

LEMMA 1. *Suppose $A, B \in \mathcal{L}(\mathcal{H})$.*
(i) *If A is one-one then $\gamma(A)\gamma(B) \leq \gamma(AB)$.*
(ii) *If BA is bounded below then $\gamma(A)\gamma(B|_{A(\mathcal{H})}) \leq \gamma(BA)$.*

Proof. If $\gamma(A) = 0$ then the statements (i) and (ii) are both evident. Thus we suppose that $\gamma(A) > 0$. For the statement (i), suppose A is one-one. Then A is bounded below, so that $\gamma(A)\,||y|| \leq ||Ay||$ for each $y \in \mathcal{H}$. Since $N(AB) = N(B)$, it follows that

$$\gamma(AB) = \inf_{x\in\mathcal{H}} \frac{||ABx||}{\text{dist}\big(x, N(AB)\big)} \geq \inf_{x\in\mathcal{H}} \frac{\gamma(A)\,||Bx||}{\text{dist}\big(x, N(B)\big)} = \gamma(A)\gamma(B),$$

which proves the statement (i). For the statement (ii), suppose BA is bounded below. Then $B|_{A(\mathcal{H})}$ is bounded below, so that $\gamma(B|_{A(\mathcal{H})})||y|| \leq ||By||$ for each $y \in A(\mathcal{H})$. Thus we have

$$\gamma(BA) = \inf_{||x||=1} ||BAx|| \geq \inf_{||x||=1} \gamma(B|_{A(\mathcal{H})})||Ax|| = \gamma(B|_{A(\mathcal{H})})\gamma(A). \quad \square$$

In Lemma 1 (ii), $\gamma(B|_{A(\mathcal{H})})$ cannot be replaced by $\gamma(B)$. For example if $A \in \mathcal{L}(\ell_2)$ is defined by $A(x_1, x_2, \cdots) = (x_1, \frac{1}{4}x_1, x_2, x_3, \cdots)$ and if U is the unilateral shift on ℓ_2, then U^*A is invertible, while $\gamma(U^*A) = \frac{1}{4}$, $\gamma(A) = 1$, $\gamma(U^*) = 1$, and $\gamma(U^*|_{A(\mathcal{H})}) = \frac{1}{\sqrt{17}}$.

COROLLARY 2. *If $A, B \in \mathcal{L}(\mathcal{H})$ is such that BA is bounded below and $N(B) \subseteq A(\mathcal{H})^\perp$, then $\gamma(A)\gamma(B) \le \gamma(BA)$.*

Proof. If $N(B) \subseteq A(\mathcal{H})^\perp$, then $\gamma(B|_{A(\mathcal{H})}) \ge \gamma(B)$. Thus the result immediately follows from Lemma 1. □

The following theorem provides a lower bound for the reduced minimum modulus of the upper triangular operator matrices with some condition on the diagonal entry.

THEOREM 3. *Suppose $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. If either B or A^* is one-one then*

$$\frac{\gamma(A)\gamma(B)}{\max\{\gamma(A), \gamma(B)\} + ||C||} \le \gamma(M_C). \tag{3.1}$$

Proof. Suppose B is one-one. We first claim that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $A_{ij} \in \mathcal{L}(\mathcal{H}_i, \mathcal{H}_j)$ for each $i, j = 1, 2$, then

$$\left|\left|\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right|\right| \le \frac{\sqrt{\alpha + 2|\beta|} + \sqrt{\alpha - 2|\beta|}}{2}, \tag{3.2}$$

where $\alpha := \sum_{i,j=1}^{2} ||A_{ij}||^2$ and $\beta := \det \begin{pmatrix} ||A_{11}|| & ||A_{12}|| \\ ||A_{21}|| & ||A_{22}|| \end{pmatrix}$. To see this, let $S := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and let $\widetilde{S} := \begin{pmatrix} ||A_{11}|| & ||A_{12}|| \\ ||A_{21}|| & ||A_{22}|| \end{pmatrix}$ be the block-norm matrix of S. Then by an argument of Hou and Du [**6**], we have that $||S|| \le ||\widetilde{S}||$. But since $||\widetilde{S}|| = ||\widetilde{S}^*\widetilde{S}||^{\frac{1}{2}} = \max\left\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } \widetilde{S}^*\widetilde{S}\right\}$, a straightforward calculation gives (3.2). Write $M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$. Note that $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is invertible for any C. But since B is one-one, we have that by Lemma 1(i),

$$\begin{aligned} \gamma(M_C) &\ge \gamma\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\gamma\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\gamma\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \\ &= \gamma\begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \cdot \min\{1, \gamma(B)\} \cdot \min\{1, \gamma(A)\}. \end{aligned} \tag{3.3}$$

On the other hand, since $\begin{pmatrix} I & -C \\ 0 & I \end{pmatrix}$ is the inverse of $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$, it follows from (3.2) that

$$\gamma \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} = \frac{1}{\left\| \begin{pmatrix} I & -C \\ 0 & I \end{pmatrix} \right\|} \geq \frac{2}{\sqrt{4 + ||C||^2} + ||C||} \geq \frac{1}{1 + ||C||},$$

which together with (3.3) gives

$$\frac{1}{1 + ||C||} \cdot \min\{1, \gamma(A), \gamma(B), \gamma(A)\gamma(B)\} \leq \gamma(M_C). \tag{3.4}$$

If one of $\gamma(A)$ and $\gamma(B)$ is zero then the inequality (3.1) is evident. Thus we assume that $\gamma(A)$ and $\gamma(B)$ are both non-zero. If $\gamma(A) \leq \gamma(B)$, then applying (3.4) with $\frac{M_C}{\gamma(B)}$ in place of M_C gives

$$\frac{1}{1 + ||\frac{C}{\gamma(B)}||} \leq \gamma\left(\frac{A}{\gamma(B)}\right) \leq \gamma\left(\frac{M_C}{\gamma(B)}\right),$$

which implies that $\frac{\gamma(A)}{1+||\frac{C}{\gamma(B)}||} \leq \gamma(M_C)$. If instead $\gamma(B) < \gamma(A)$, then the same argument with $\frac{M_C}{\gamma(A)}$ in place of $\frac{M_C}{\gamma(B)}$ gives that $\frac{\gamma(B)}{1+||\frac{C}{\gamma(A)}||} \leq \gamma(M_C)$. This gives (3.1). If A^* is one-one then the same argument with M_C^* in place of M_C proves (3.1). □

COROLLARY 4. *Suppose* $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. *If A has dense range and B is one-one then*

$$\frac{\gamma(A)\gamma(B)}{\max\{\gamma(A), \gamma(B)\} + ||C||} \leq \gamma(M_C) \leq \min\{\gamma(A), \gamma(B)\}. \tag{4.1}$$

Proof. If B is one-one then $N(M_C) = N(A) \oplus \{0\}$, so that evidently $\gamma(M_C) \leq \gamma(A)$. If A has dense range then the inequality $\gamma(M_C) \leq \gamma(B)$ is known from [**1**, Lemma 1.3]. This proves the upper bound. The lower bound comes from (3.1). □

Corollary 5. *Suppose* $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. *If* $R(C) \subseteq R(A)^\perp$ *then*

$$\gamma(M_C) \geq \min\{\gamma(A),\, \gamma(B)\} = \gamma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{5.1}$$

Proof. Since by assumption $A^*C = 0$, we have

$$|M_C| = \begin{pmatrix} |A| & 0 \\ 0 & (|C|^2+|B|^2)^{\frac{1}{2}} \end{pmatrix},$$

which implies

$$\begin{aligned}
\gamma(M_C) &= \inf\left(\sigma(|M_C|) \setminus \{0\}\right) \\
&= \inf\left([\sigma(|A|) \cup \sigma((|C|^2 + |B|^2)^{\frac{1}{2}})] \setminus \{0\}\right) \\
&\geq \inf\left([\sigma(|A|) \cup \sigma(|B|)] \setminus \{0\}\right) \\
&= \min\{\gamma(A),\, \gamma(B)\} = \gamma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.
\end{aligned}$$

□

Corollary 6. *Suppose* $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. *If A has dense range and B is one-one then M_C has closed range if and only if both A and B have closed ranges.*

Proof. Straightforward from Corollary 4. □

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