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# A CHARACTERIZATION OF NEARLY SIGN-CENTRAL MATRICES

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Abstract. The sign-central matrices were characterized by Ando and Brualdi. In this paper, we define a nearly sign-central matrices and give a characterization of nearly sign-central matrices.

## 1. Introduction

The *sign* of a real number $a$ is defined to be

$$\operatorname{sign} a = \begin{cases} 0 & \text{if } a = 0, \\ +1 & \text{if } a > 0, \\ -1 & \text{if } a < 0. \end{cases}$$

The *sign pattern* of a real matrix $A$ is the $(0,1,-1)$-matrix obtained from $A$ by replacing each entry by its sign. A real matrix, $A$, determines a *qualitative class*, $\mathcal{Q}(A)$, consisting of all matrices with the same sign pattern as $A$. The column vectors $b^{(1)},\dots,b^{(n)}$ of a matrix $B$ in $\mathcal{Q}(A)$ determine a convex polytope

$$\mathcal{C}(B) = \Big\{ \sum_{i=1}^{n} c_i b^{(i)} \mid \sum_{i=1}^{n} c_i = 1,\ \ c_i \ge 0,\ \ 1 \le i \le n \Big\}.$$

We define a real matrix $A$ to be *central* provided that the origin is in the convex hull of its columns. It is well known that central matrices arise in many applications and in linear programming problems.

In [1,3] a qualitative analog, which we now describe, of centrality is studied. A $(0,1,-1)$-matrix $A$ is *sign-central* provided that every

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matrix in $Q(A)$ is central. A matrix is *minimal sign-central* if it is sign-central and no matrix obtained by deleting one of its columns is sign-central. Sign-central and minimal sign-central matrices have been studied in [1,3], and in [6] where they play a crucial role in the study of sign-solvable linear systems of the form $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$ (here, $\mathbf{x} \geq 0$ means $\mathbf{x}$ is nonzero and entrywise nonnegative). Also, a sign-central linear preserver was studied in [5].

In investigating sign-central matrices there is no loss in generality in restricting attention to $(0, 1, -1)$-matrices. This is because a matrix $A$ is a sign-central matrix if and only if each matrix in $Q(A)$ is sign-central. For example, the $m \times (m+1)$ matrix with exactly one 1 and exactly one $-1$ in each row defined by

$$F_m = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

is easily seem to be sign-central matrix.

Let $A$ be an $m \times n$ sign pattern matrix, $m \leq n$. Then, clearly, the matrix $A$ is either sign-central or not. If the matrix $A$ is not sign-central, then we have two cases as followings: the first case is that there is no nonzero sign pattern vector $\alpha$ such that $[A \vdots \alpha]$ is a sign-central matrix, and the second case is that there is a nonzero sign pattern vector $\alpha$ such that $[A \vdots \alpha]$ is a sign-central matrix. A matrix $A$ satisfying the latter condition are called *nearly sign-central* matrix.

For example,

$$(1) \qquad A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

is not a sign-central matrix, but

$$A' = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad A'' = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

are sign-central matrices. Thus, the matrix $A$ is a nearly sign-central matrix.

For a matrix $A$, $A(\cdot|j)$ will denote the matrix obtained from $A$ by striking out the $j$th column of $A$.

If $A$ is a nearly sign-central matrix, then there is a nonzero vector $\alpha$ such that $[A \vdots \alpha]$ is a sign-central matrix. But, for some nearly sign-central matrix $A$, $[A \vdots \alpha]$ may not be a minimal sign-central matrix. In the above example, the matrix $A'$ is a minimal sign-central matrix, but $A''$ is not minimal sign-central matrix since $A''(\cdot|3)$ is a sign-central matrix. In general, if $A$ is a nearly sign-central matrix, then $A' = [A \vdots \alpha]$ is not always a minimal sign-central matrix. Clearly, if $A$ is a minimal sign-central matrix, then $A(\cdot|k)$ is a nearly sign-central matrix for each $k$.

A diagonal matrix $D \neq 0$ each of whose diagonal entries equals 0, 1, or $-1$ is called a *signing*. A signing with no 0's on its main diagonal is called a *strict signing*. Let $A$ be an $m \times n$ matrix, and let $P$ and $Q$ be permutation matrices of order $m$ and $n$, respectively. Let $D$ be a strict signing. Then it follows from the definition that $A$ is a (minimal) sign-central matrix if and only if $PDAQ$ is a (minimal) sign-central matrix. Now, assume that $A$ has a zero row, and let $A'$ be a matrix obtaining from $A$ by deleting a zero row. Then $A$ is a (minimal) sign-central matrix if and only if $A'$ is a (minimal) sign-central matrix. This observation implies that we may assume that the matrices considered do not have zero rows. Similarly, $A$ is a nearly sign-central matrix if and only if $PDAQ$ is a nearly sign-central. Also, wé may assume that the nearly sign-central matrices have no zero rows.

In this paper, we study nearly sign-central matrices and we give a characterization of such matrices.

## 2. Nearly Sign-Central Matrices

We begin this section by a result from [1]. The following characterization of sign-central matrices is contained in theorem 2.1 in [1].

THEOREM 2.1 ([1]). *Let $A$ be an $m \times n$ $(0, 1, -1)$-matrix. Then the following are equivalent:*

(i) *$A$ is a sign-central matrix,*

(ii) *for each strict signing $D$ of order $m$, there exists a nonnegative column vector of $DA$,*

(iii) *for each strict signing $D$ of order $m$, there exists a nonpositive column vector of $DA$.*

By the above theorem, a matrix $A$ is sign-central if and only if for each strict signing $D$ of order $m$, the matrix $DA$ have both a nonnegative column vector and a nonpositive column vector. Clearly, if $D$ is a strict signing matrix, then $-D$ is also strict signing. Hence $-DA$ have both a nonnegative column vector and a nonpositive column vector. So, we identify a strict signing $-D$ with $D$.

LEMMA 2.2. *Let $A$ be an $m \times n$ nearly sign-central matrix. Then there exists a strict signing $D$ of order $m$ such that $DA$ has a nonnegative column vector and does not have a nonpositive column vector.*

*Proof.* Since $A$ is a nearly sign-central matrix, there exists a nonzero vector $\alpha$ such that the matrix $A' = [A \vdots \alpha]$ is a sign-central matrix. So, for each strict signing $D$ of order $m$, $DA'$ have both a nonnegative and a nonpositive column vector. If $DA$ have both a nonnegative column vector and a nonpositive column vector for every strict signing $D$, then $A$ is a sign-central and hence this is a contradiction. If $DA$ have no nonnegative and nonpositive column vector for some $D$, then $A$ is not a nearly sign-central matrix. Thus, there exists a strict signing $D$ such that $DA$ has a nonnegative column vector and does not have a nonpositive column vector, which completes the proof. □

Let $\mathcal{T} = \{0, 1, -1\}$. We now introduce sign-subconformality in $\mathcal{T}$. For $a, b \in \mathcal{T}$, we say that $a$ is *sign-subconformal* to $b$, denoted by $a \sqsubseteq b$, provided $a \in \{0, b\}$. For vectors $\alpha = (\alpha_1, \cdots, \alpha_n)^T$, $\beta = (\beta_1, \cdots, \beta_n)^T \in \mathcal{T}^n$, we define $\alpha \sqsubseteq \beta$ to be

$$\alpha \sqsubseteq \beta \text{ if and only if } \alpha_i \sqsubseteq \beta_i \text{ for each } i = 1, \cdots, n.$$

For example

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \sqsubseteq \begin{bmatrix} 1 \\ -1 \\ \pm 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \not\sqsubseteq \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \not\sqsubseteq \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Clearly, $\alpha \sqsubseteq \beta$ if and only if $D\alpha \sqsubseteq D\beta$ for every strict signing $D$. In [4], a sign-subconformal linear preserver was studied.

THEOREM 2.3. *Let $A$ be an $m \times n$ nearly sign-central matrix. If $[A \vdots \alpha]$ is a sign-central matrix for some nonzero vector $\alpha$, then $[A \vdots \alpha']$ is a sign-central matrix for all nonzero vector $\alpha'$ with $\alpha' \sqsubseteq \alpha$.*

*Proof.* Let $X$ be the set of all strict signings $D$ of order $m$ such that $DA$ has a nonnegative column vector and does not have a nonpositive column vector. Then, by lemma 2.2, the set $X$ is not empty. Since $[A \vdots \alpha]$ is a sign-central matrix, $D\alpha$ is a nonpositive vector for each $D \in X$. Since $\alpha$ is a nonzero vector, there exists a nonzero vector $\alpha'$ such that $\alpha' \sqsubseteq \alpha$. And, for every strict signing $S$ of order $m$, $S\alpha' \sqsubseteq S\alpha$. Thus, $D\alpha'$ is a nonpositive vector for each $D \in X$, which implies $[A \vdots \alpha']$ is a sign-central matrix. □

The following is an immediate consequence of Theorem 2.3.

COROLLARY 2.4. *Let $A$ be a nearly sign-central matrix. Then there exists a vector $\alpha$ with one nonzero entry such that $[A \vdots \alpha]$ is a sign-central matrix.*

For example, there is a vector $\alpha = (0, 0, -1)^T$ or $\alpha = (-1, 0, 0)^T$ such that $[A \vdots \alpha]$ is a sign-central matrix for the nearly sign-central matrix $A$ in (1). Corollary 2.4 is very useful result for the study of nearly sign-centrality and the following is an immediate consequence of Corollary 2.4.

COROLLARY 2.5. *Let $A = [a^{(1)} \cdots a^{(n)}]$ be an $m \times n$ nearly sign-central matrix. If $\alpha$ is a vector with exactly one nonzero entry such that $[A \vdots \alpha]$ is a sign-central matrix, then $\alpha \sqsubseteq -a^{(i)}$ for some $i$ with $1 \leq i \leq n$.*

For example, the matrix $A$ in (1) is a nearly sign-central and

$$[A \vdots \alpha] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

is a sign-central matrix. Thus there is a vector $(0, -1, 1)^T$ in $A$ such that $\alpha = (0, 0, -1)^T \sqsubseteq -(0, -1, 1)^T$.

THEOREM 2.6. *Let $A = [a^{(1)} \cdots a^{(n)}]$ be an $m \times n$ nearly sign-central matrix. If there exist $k$ and $l$, $1 \leq k \neq l \leq n$, such that $Da^{(l)} \sqsubseteq Da^{(k)}$ for each strict signing $D$, then $A(\cdot|k)$ is a nearly sign-central matrix.*

*Proof.* Suppose that there exist $k$ and $l$, $1 \leq k \neq l \leq n$ such that $Da^{(l)} \sqsubseteq Da^{(k)}$ for each strict signing $D$. Let $d_i$ be the number of nonzero entries of $a^{(i)}$, $i = 1, \ldots, n$. Then, $d_l \leq d_k$. If, for some strict signing $D$, $Da^{(k)}$ is a nonnegative(nonpositive, resp.) vector, then $Da^{(l)}$ is also nonnegative(nonpositive, resp.) vector. Since $A$ is a nearly sign-central, there is a nonzero vector $\alpha$ such that $[A \vdots \alpha]$ is a sign-central. Thus, for every strict signing $D$, $D[A(\cdot|k) \vdots \alpha]$ have both a nonnegative and a nonpositive column vector. Therefore, $A(\cdot|k)$ is a nearly sign-central matrix. □

Suppose that

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Then the matrix $A$ is a nearly sign-central matrix which of each column does not have a sign-subconformality. Since $A(\cdot|4)$ is a nearly sign-central matrix, the converse of Corollary 2.5 is not true, in general.

Now, we give a characterization of nearly sign-central matrices.

THEOREM 2.7. *Let $A = [a^{(1)} \cdots a^{(n)}]$ be an $m \times n$ matrix which is not a sign-central. Then $A$ is a nearly sign-central matrix if and only if there exists permutation matrices $P$ and $Q$ such that*

$$PAQ = \begin{bmatrix} 0 \cdots 0\, 1 \cdots 1 & \vdots & -1 \cdots -1 \\ \cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ A_1 & \vdots & A_2 \end{bmatrix}$$

*or*

$$PAQ = \begin{bmatrix} 0 \cdots 0\, -1 \cdots -1 & \vdots & 1 \cdots 1 \\ \cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ A_1 & \vdots & A_2 \end{bmatrix},$$

*where $A_1$ is an $(m-1) \times p$ sign-central matrix, $1 \le p \le n$, and $A_2$ is an $(m-1) \times (n-p)$ $(0,1,-1)$-matrix.*

*Proof.* Suppose that $A$ is a nearly sign-central matrix. Since nearly sign-central matrices are permutation invariant, we may assume that $A$ has one of the following forms:

$$\begin{bmatrix} 0 \cdots 0\ 1 \cdots 1 & \vdots & -1 \cdots -1 \\ \cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ A_1 & \vdots & A_2 \end{bmatrix}, \tag{2}$$

or

$$\begin{bmatrix} 0 \cdots 0\ -1 \cdots -1 & \vdots & 1 \cdots 1 \\ \cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ A_1 & \vdots & A_2 \end{bmatrix}. \tag{3}$$

First, assume the case of the form (2). By Corollary 2.4, without loss of generality, we may assume that $A' = [A \vdots \alpha]$ is a sign-central, where $\alpha = (-1, 0, \dots, 0)^T$.

Now, we will show that $A_1$ is a sign-central matrix. For any strict signing $\tilde{D}$ of order $m-1$, let

$$\mathcal{S}_1 = \{D_1 | D_1 = [1] \oplus \tilde{D}\} \text{ and } \mathcal{S}_2 = \{D_2 | D_2 = [-1] \oplus \tilde{D}\}.$$

Note that for each $D_1 \in \mathcal{S}_1$, $D_1 A'$ have both a nonnegative column vector and a nonpositive column vector. Since, for each $D_1$, $D_1\alpha$ is a nonpositive vector, there is a nonpositive vector in $\{D_1 a^{(1)}, \dots, D_1 a^{(p)}\}$. Thus, for every strict signing $\tilde{D}$ of order $m-1$, $\tilde{D}A_1$ has a nonnegative column vector. Therefore, $A_1$ is a sign-central matrix by Theorem 2.1. By similar method, we can show one for the case of the form (3).

Conversely, suppose that $A$ has one of the forms (2) and (3) up to permutations. First, assume that $A$ has the form (2). For a vector $\alpha = (-1, 0, \dots, 0)^T$, let $A' = [A \vdots \alpha]$. Then, for each $D_1 \in \mathcal{S}_1$, $D_1\alpha$ is a nonpositive vector. Since $A_1$ is a sign-central matrix, there exists $k$ with $1 \le k \le p$ such that $D_1 a^{(k)}$ is a nonnegative column vector. Since, for each $D_2 \in \mathcal{S}_2$, $D_2\alpha$ is a nonnegative vector and $A_1$ is a sign-central matrix, there exists $l$ with $1 \leq l \leq p$ such that $D_2 a^{(l)}$ is a nonpositive column vector. Thus, for every strict signing $D$ of order $m$, $DA'$ have both a nonnegative column vector and a nonpositive column vector.

In the case (3), the proof is similar to the proof of the case (2). Thus, the proof is completed. □

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