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STABILITY THEOREMS OF THE OPERATOR-VALUED FUNCTION SPACE INTEGRAL ON $C_0(B)$

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ABSTRACT. In 1968, Cameron and Storvick introduce the definition and the theories of the operator-valued function space integral. Since then, the stability theorems of the integral was developed by Johnson, Skoug, Chang etc [1, 2, 4, 5]. Recently, the authors establish the existence theorem of the operator-valued function space [8].

In this paper, we will prove the stability theorems of the operator-valued function space integral over paths in abstract Wiener space $C_0(\mathbf{B})$.

1. Preliminaries

In this section, we describe some notations, definitions and known facts which will be needed in the subsequent sections.

Let $(\mathbf{B}, B(\mathbf{B}), m)$ be an abstract Wiener space. For $\lambda > 0$, let m_λ be the Borel measure on $\mathbf{B}$ given by m_λ (B) $= m(\lambda^{-1}\text{B})$ for Borel subset B of $\mathbf{B}$. Let $C(\mathbf{B})$ denote the set of all $\mathbf{B}$-valued continuous functions on $[a, b]$ and let $C_0(\mathbf{B})$ denote the set of all continuous functions on $[a, b]$ which vanish at a. Then $C_0(\mathbf{B})$ is a real separable Banach space in the norm $\| y \|_{C_0(\mathbf{B})} \equiv \sup_{a \le t \le b} \| y(t) \|_{\mathbf{B}}$. For y in $C(\mathbf{B})$, y has the unique decomposition $y = x + \xi$, where x is in $C_0(\mathbf{B})$ and ξ is in $\mathbf{B}$. Then Brownian motion in $\mathbf{B}$ induces a probability measure $m_{\mathbf{B}}$ on $(C_0(\mathbf{B}), B(C_0(\mathbf{B})))$ which is mean-zero Gaussian, as following ; let

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$\overrightarrow{t} = (t_1, t_2, \cdots, t_n)$ be given with $a = t_0 < t_1 < t_2, \cdots, t_n \le b$. Let $T_{\overrightarrow{t}} : \mathbf{B}^n \to \mathbf{B}^n$ be given by

$$
\begin{aligned}
(1.1)\quad & T_{\overrightarrow{t}}(x_1, x_2, \cdots, x_n) \\
&= (\sqrt{t_1 - t_0}\, x_1, \sqrt{t_1 - t_0}\, x_1 + \sqrt{t_2 - t_1}\, x_2, \cdots, \sum_{i=1}^{n} \sqrt{t_i - t_{i-1}}\, x_i).
\end{aligned}
$$

Then we can find that $m_{\mathbf{B}}$ is well defined, countable additive, mean zero, stationary increment, and Gaussian measure.

By the change of variable theorem, we have

LEMMA 1.1 (WIENER INTEGRATION THEOREM). *Let $\overrightarrow{t} = (t_1, t_2, \cdots, t_n)$ be given with $a = t_0 < t_1 < t_2, \cdots, t_n \le b$ and $f : \mathbf{B}^n \to \mathbf{C}$ be a* Borel *measurable function. Then*

$$
\begin{aligned}
(1.2)\qquad & \int_{C_0(\mathbf{B})} f(y(t_1), y(t_2), \cdots, y(t_n))\, dm_{\mathbf{B}}(y) \\
& \stackrel{*}{=} \int_{\mathbf{B}^n} f \circ T_{\overrightarrow{t}}(x_1, x_2, \cdots, x_n)\, d\Big(\prod^{n} m\Big)(x_1, x_2, \cdots, x_n),
\end{aligned}
$$

where by $\stackrel{*}{=}$, we mean that if either side exists, both sides exist and they are equal.

In [3], Chung considered the Borel subsets Ω_λ, $\lambda > 0$ and D of an abstract Wiener space $\mathbf{B}$ which satisfies the following ; for two positive reals, λ_1 and λ_2, $\lambda_1 \Omega_{\lambda_2} = \Omega_{\lambda_1 \lambda_2}$ and $\mathbf{B}$ is the disjoint union of this family of sets. Also $m(\Omega_\lambda) = 0$ if and only if $\lambda \ne 1$. Let $(\mathbf{B}, \bar{B}(\mathbf{B}), \bar{m})$ be the completion of $(\mathbf{B}, B(\mathbf{B}), m)$. A subset N of $\mathbf{B}$ is called the scale-invariant null subset (s-null set) provided that for all $\lambda > 0$, $m_\lambda(N) = 0$. A subset S of $\mathbf{B}$ is called the scale-invariant measurable subset provided that for $\lambda > 0$, there is a m_λ-measurable subset S_λ of Ω_λ such that $S = \big(\bigcup_{\lambda > 0} S_\lambda\big) \cup D$ where D is a subset of $\mathbf{B} \setminus \bigcup_{\lambda > 0} \Omega_\lambda$. And, we say that the propositional function $p(x)$ on $\mathbf{B}$ holds s-a.e. if the set $\{x \mid p\ (x)$ does not true$\}$ is an s-null set.

DEFINITION 1.2. Let $L_{p,\infty}(\mathbf{B})$ $(1 \le p < \infty)$ be the class of all $\mathbf{C}$-valued Borel measurable function ψ on $\mathbf{B}$ such that for $\lambda > 0$, $\mid \psi(\lambda(\cdot)) \mid^p$ is

m-integrable and

$$(1.3) \qquad \| \psi \|_{p,\infty} \equiv \sup_{\lambda > 0} \| \psi(\lambda(\cdot)) \|_p = \sup_{\lambda > 0} \Big[\int_{\mathbf{B}} | \psi(\lambda x) |^p \, dm(x) \Big]^{\frac{1}{p}}$$

is finite. For f and g in $L_{p,\infty}(\mathbf{B})$, we say that f is equivalent to g, denote $f \sim g$ if $\{\lambda x \in \mathbf{B} \mid f(x) \neq g(x)\}$ is an m_λ-null set for all $\lambda > 0$. Clearly $\sim$ is an equivalent relation on $L_{p,\infty}(\mathbf{B})$. Hence we obtain a quotient space $L_{p,\infty}(\mathbf{B})/\sim$ which we denote $\mathcal{L}_{p,\infty}(\mathbf{B})$. From [8], we have $(\mathcal{L}_{p,\infty}(\mathbf{B}), \| \cdot \|_{p,\infty})$ is a Banach space.

DEFINITION 1.3. For $\lambda > 0$, we define an operator $\mathcal{C}_\lambda$ on $\mathcal{L}_{p,\infty}(\mathbf{B})$ given by

$$(1.4) \qquad (\mathcal{C}_\lambda \psi)(x) = \int_{\mathbf{B}} \psi(\lambda^{-\frac{1}{2}} x_1 + x) \, dm(x_1)$$

for ψ in $\mathcal{L}_{p,\infty}(\mathbf{B})$.

By the above definition, we easily check that for $\lambda > 0$, $\mathcal{C}_\lambda$ is bounded linear operator from $\mathcal{L}_{p,\infty}(\mathbf{B})$ into itself, and $||| \, \mathcal{C}_\lambda \, ||| \leq 1$.

DEFINITION 1.4. Let $\theta : [a,b] \times \mathbf{B} \to \mathbf{C}$ be a bounded Borel measurable function. We define the multiplication operator $M_{\theta(s,\cdot)}$ by $\big(M_{\theta(s,\cdot)} \psi\big)(x) = \theta(s,x)\psi(x)$. Let $\theta(s)$ denote the operator $M_{\theta(s,\cdot)}$ of multiplication by $\theta(s,\cdot)$ acting in $\mathcal{L}_{p,\infty}(\mathbf{B})$.

REMARK. In the above definition 1.4, $\theta(s)$ is a bounded linear operator from $\mathcal{L}_{p,\infty}(\mathbf{B})$ into itself and $||| \, \theta(s) \, ||| \leq \sup_{x \in \mathbf{B}} | \theta(s,x) |$.

DEFINITION 1.5. Let $F : C(\mathbf{B}) \to \mathbf{C}$ be a function, let $\lambda > 0$ be given, let ψ be in $\mathcal{L}_{p,\infty}(\mathbf{B})$ and let x be in $\mathbf{B}$. We define

$$(1.5) \qquad \big[K_\lambda(F)\psi\big](x) = \int_{C_0(\mathbf{B})} F(\lambda^{-\frac{1}{2}} y + x) \psi(\lambda^{-\frac{1}{2}} y(b) + x) \, dm_{\mathbf{B}}(y).$$

If $K_\lambda(F)$ exists and $K_\lambda(F)$ is a bounded linear operator from $\mathcal{L}_{p,\infty}(\mathbf{B})$ into itself for each $\lambda > 0$. We say that the operator-valued function space integral $K_\lambda(F)$ exists for all $\lambda > 0$.

We adopt the following notations and assumptions throughout this paper. Let $\theta : [a,b] \times \mathbf{B} \to \mathbf{C}$ be a bounded Borel measurable function by the upper bound M and let η be a $\mathbf{C}$-valued Borel measure on (a,b). $\eta = \mu + \sigma$ will be the decomposition of η into its continuous part μ and discrete part $\sigma = \sum_{p=1}^{h} \omega_p \delta_{\tau_p}$. And let

$$F_n(y) = \Big(\int_{(a,b)} \theta(s, y(s))\, d\eta(s) \Big)^n \tag{1.6}$$

for y in $C_0(\mathbf{B})$. Let δ_{τ_p}be the Dirac measure with total mass one concentrated at τ_p.

DEFINITION 1.6. Let (Ω, μ) be a measure space and let $f : \Omega \to \mathcal{L}(\mathcal{L}_{p,\infty}(\mathbf{B}))$,the space of all bounded linear operator from $\mathcal{L}_{p,\infty}(\mathbf{B})$ to itself, be a function. We say that f is (s-w)-integrable if there exists $U \in \mathcal{L}(\mathcal{L}_{p,\infty}(\mathbf{B}))$ such that for ψ in $\mathcal{L}_{p,\infty}(\mathbf{B})$, $\varphi \in \mathcal{L}_{q,\infty}(\mathbf{B})$, ν a Borel measure on $(0, +\infty)$, $\lambda > 0$,

$$\begin{aligned} &\int_{(0,+\infty)} \int_{\Omega_\lambda} \big[U\psi\big](x)\varphi(x)\, dm_\lambda(x)\, d\nu(\lambda) \\ &= \int_\Omega \int_{(0,+\infty)} \int_{\Omega_\lambda} \big[f(\omega)\psi\big](x)\varphi(x)\, dm_\lambda(x)\, d\nu(\lambda)\, d\mu(\omega). \end{aligned} \tag{1.7}$$

In this case, we write $U = (s-w) - \int_\Omega f(\omega) d\mu(\omega)$.

REMARK. We easily check that (s-w)-integral is well defined [8]. The conditions of (s-w)-integrable is rather weaker than the Bochner integral for the operator-valued function.
From [8], we have following facts.

THEOREM 1.7. *If f is (s-w)-integrable on Ω, f is bounded and A is a measurable subset of (Ω, μ), then for ψ in $\mathcal{L}_{p,\infty}(\mathbf{B})$, f is* (s-w)-*integrable on A. Moreover*

$$\Big[(s-w) - \int_A f(\omega)\, d\mu(\omega)\Big]\psi(x) = \int_A \Big[f(\omega)\psi\Big](x)\, d\mu(\omega) \quad \text{s} - \text{a.e. } x.$$

THEOREM 1.8. *Let f be* (s-w)*-integrable on Ω such that $|||\, f\,|||$ is bounded. Then*

$$(1.8)\qquad \|\,(s-w)-\int_\Omega f(\omega)\,d\mu(\omega)\,\| \le \;|||\, f\,|||_\infty \mid \mu \mid (\Omega).$$

THEOREM 1.9. *Let $F_n(x) = \left(\int_{(a,b)} \theta(s,x(s)),\ d\eta(s)\right)^n$ and θ be bounded by M, where η be a* Borel *measure on (a,b) and μ be a continuous part of η, $\sigma = \sum_{p=1}^h w_i\delta_{\tau_p}$ be a discrete part of η. Then for any $\lambda > 0$, there is an operator-valued function space integral $K_\lambda(F_n)$ of F_n such that*

$$(1.9)\qquad \begin{aligned} &K_\lambda(F_n)\\ &= \sum_{q_0+\cdots+q_h=n} n!\frac{\omega_1^{q_1}\cdots\omega_h^{q_h}}{q_1!\cdots q_h!} \sum_{j_1+\cdots+j_{h-1}=q_0} (s-w)\\ &\quad -\int_{\triangle_{q_0;j_1,\cdots,j_{h-1}}} \Big[L_0\circ L_1\circ\cdots\circ L_h(s_1,s_2,\cdots,s_h)\Big]\, d\left(\prod_{p=1}^{h+1}\prod_{i=1}^{j_p}\mu\right)(s_{p-1,i}), \end{aligned}$$

where for $k = 0,1,2,\cdots,h$,

$$L_k = C_{\alpha_{k-1,1}}\circ\theta(s_{k,1})\circ C_{\alpha_{k-1,2}}\circ\theta(s_{k,2})\circ\cdots\circ\theta(s_{k,j_{k-1}})\circ\Big\{\theta(\tau_{k+1})\Big\}^{q_{k+1}}.$$

$$\begin{aligned}\triangle_{q_0;j_1,\cdots,j_{h-1}} = &\Big\{(s_{0,1},s_{0,2},\cdots,s_{q_{0,j_1}},s_{1,1},s_{1,2},\cdots,s_{h,j_{h-1}}) \;\big|\; a=s_{0,0}<\\ &s_{0,1}<\cdots<s_{0,j_1}<\tau_1<s_{1,1}<\cdots<s_{1,2}<\cdots<\\ &s_{h-1,j_h}<\tau_h<s_{h,1}<\cdots<s_{h,j_{h-1}}<b=\tau_{h+1}\Big\}\end{aligned}$$

for $p = 1,2,\cdots,h+1$ and for $i = 1,2,\cdots,j_p$,

$$\begin{aligned}\alpha_{p,i} &= \lambda/(s_{p-1,i}-s_{p-1,i-1}),\\ \alpha_{p,j_p+1} &= \lambda/(s_{p,0}-s_{p-1,j_p}).\end{aligned}$$

Moreover, $|||\, K_\lambda(F_n)\,||| \le (M \mid \eta \mid)^n$.

2. The main theorems

In this section, we will prove the stability theorem for the operator-valued function space integral over paths in abstract Wiener space.

DEFINITION 2.1. Let $< \theta_n >$ be a bounded sequence of the complex-valued Borel measurable function on $\mathbf{B}$. We say that $< \theta_n >$ converges uniformly s-a.e. if there is a complex-valued function θ on $\mathbf{B}$ such that for each $\epsilon > 0$, $\lim_{n\to 0} m_\lambda\{x \ : \ |\theta_n(x) - \theta(x)| > \epsilon\} \ = \ 0$ uniformly for $\lambda > 0$.

LEMMA 2.2. *In the above, a sequence $< M_{\theta_n} >$ converges to M_θ as $n \to \infty$ in the operator-norm topology.*

Proof. Suppose a sequence $< \theta_n >$ is bounded by M and ψ is in $\mathcal{L}_{p,\infty}(\mathbf{B})$ with $\| \psi \|_{p,\infty} = 1$. Let $\epsilon > 0$ be given. Then for $\lambda > 0$, there is a $\delta > 0$ such that for every Borel subsets K_λ of Ω_λ with $m_\lambda(K_\lambda) < \delta$,

$$\int_{K_\lambda} | \psi(x) |^p \, dm_\lambda(x) \ < \ \epsilon \tag{2.1}$$

which implies that for any scale-invariant measurable subset K with $m_\lambda(K) < \delta$ for all $\lambda > 0$, $\int_K | \psi(x) |^p \, dm_\lambda(x) \ < \ \epsilon$ for all $\lambda > 0$, for example $K = \bigcup_{\lambda>0} K_\lambda$.

Since $< \theta_n >$ converges to θ uniformly s-a.e., there is an n_0 in $\mathbf{N}$ such that $n \geq n_0$ implies $m_\lambda\{x \in \mathbf{B} | \ |\theta_\mathbf{n}(\mathbf{x}) - \theta(\mathbf{x})| > \epsilon\} \ < \ \delta$ for all $\lambda > 0$. Let $T_n \ = \ \{x \in \mathbf{B} | \ |\theta_\mathbf{n}(\mathbf{x}) - \theta(\mathbf{x})| > \epsilon\}$ for $n \in \mathbf{N}$. Then T_n is a scale-invariant measurable subset and for $n \geq n_0, \ m_\lambda(T_n) < \delta$ for all $\lambda > 0$, and so for $\| \psi \|_{p,\infty} = \ 1$,

$$\begin{aligned}
&\sup_{\lambda>0}\Big[\int_{\Omega_\lambda} | \ M_{\theta_n}\psi(x) \ - \ M_\theta\psi(x) \ |^p \ dm_\lambda(x)\Big]^{\frac{1}{p}} \\
&= \sup_{\lambda>0}\Big[\int_{\Omega_\lambda - T_n} | \ \theta_n(x) \ - \ \theta(x) \ |^p \ | \ \psi(x) \ |^p \ dm_\lambda(x) \\
&\quad + \int_{T_n} | \ \theta_n(x) \ - \ \theta(x) \ |^p \ | \ \psi(x) \ |^p \ dm_\lambda(x)\Big]^{\frac{1}{p}} \\
&\leq \sup_{\lambda>0} \Big[\epsilon^p \int_{\Omega_\lambda} | \ \psi(x) \ |^p \ dm_\lambda(x) \\
&\quad + (2M)^p \int_{T_n} | \ \psi(x) \ |^p \ dm_\lambda(x)\Big]^{\frac{1}{p}}
\end{aligned} \tag{2.2}$$

$$\leq \sup_{\lambda>0} \left[\epsilon^p \int_{\Omega_\lambda} \mid \psi(x) \mid^p \ dm_\lambda(x) + (2M)^p \epsilon\right]^{\frac{1}{p}}$$
$$\leq \epsilon \ + \ 2M\epsilon^{\frac{1}{p}}.$$

Hence for $n \geq n_0$,

$$\sup_{\|\psi\|_{p,\infty}=1} \| (M_{\theta_n} \ - \ M_\theta)\psi \|_{p,\infty} \leq \ \epsilon \ + \ 2M\epsilon^{\frac{1}{p}} \ , \ \text{ as desired.} \qquad \square$$

THEOREM 2.3. *Let $F_n^m(x) = \left(\int_a^b \theta_m(s, x(s))\, d\eta(s)\right)^n$ as a given in* Theorem 1.9. *Suppose that for all $n \in \mathbf{N}$, θ_m is uniform bounded* Borel *measurable on $[a,b]\times\mathbf{B}$ and there is a bounded* Borel *measurable function θ from $[a,b]\times\mathbf{B}$ to $\mathbf{C}$ such that for η s-a.e., $< \theta_m(s,\cdot) >$ converges to $\theta(s,\cdot)$ uniformly.*
Then $K_\lambda(F_n^m) \to K_\lambda(F_n)$ in the operator norm topology.

Proof. Let $L_k^{(m)} = \mathcal{C}_{\alpha_{k+1,1}} \circ \theta_m(s_{k,1}) \circ \mathcal{C}_{\alpha_{k+1,2}} \circ \theta_m(s_{k,2}) \circ \cdots \circ \mathcal{C}_{\alpha_{k+1,j_{k+1}}} \circ \theta_m(s_{k,j_{k-1}}) \circ \left\{\theta_m(\tau_{k+1})\right\}^{q_{k-1}}$ for $m \in \mathbf{N}$ and for $k = 0,1,2,\cdots,h$ where $(s_{0,1},\cdots,s_{h,j_{h-1}})$ is in $\triangle_{q_0;j_1,\cdots,j_{h-1}}$ in Theorem 1.9 and $\left[\theta_m(\tau_{h+1})\right]^{q_{h+1}} = \left[\theta(\tau_{h+1})\right]^{q_{h-1}} = I$, an identity map.

Let $\left\{ \| \theta_m \|_\infty | \ m \in \mathbf{N}\right\} \cup \left\{ \ \| \theta \| \ \right\}$ be bounded by M, let $U = \max\left\{\mathrm{M}, 1\right\}$. For $u = 0,1,\cdots,h, \ \ v = 1,2,\cdots,j_{u+1}$, let

$$\begin{aligned} A_{u,v}^{(m)} \ &= \ L_0^{(m)} \circ L_1^{(m)} \circ \cdots \circ L_{u-1}^{(m)} \circ \mathcal{C}_{\alpha_{u+1,1}} \circ \theta_m(s_{u,1}) \circ \cdots \circ \mathcal{C}_{\alpha_{u-1,v-1}} \\ &\circ \theta_m(s_{u,v-1}) \circ \mathcal{C}_{\alpha_{u-1,v}} \circ \theta(s_{u,v}) \circ \cdots \circ \mathcal{C}_{\alpha_{u-1,j_{u-1}}} \\ &\circ \theta(s_{u,j_{u-1}}) \circ (\theta(\tau_{j_{u-1}}))^{q_{u-1}} \circ L_{u+1} \circ \cdots \circ L_h, \end{aligned}$$

and

$$A_{u+1,0}^{(m)} \ = \ L_0^{(m)} \circ L_1^{(m)} \circ \cdots \circ L_{u-1}^{(m)} \circ L_u^{(m)} \circ L_{u+1} \circ \cdots \circ L_h^{(m)}.$$

Then, by Theorem 1.8 and Theorem 1.9,

$$\begin{aligned}
&\||| L_0^{(m)} \circ L_1^{(m)} \circ \cdots \circ L_h^{(m)} - L_0 \circ L_1 \circ \cdots \circ L_h \||| \\
&\leq \sum_{u=0}^{h} \Big[\Big\{ \sum_{v=1}^{j_{u+1}} \||| A_{u,v}^{(m)} - A_{u,v-1}^{(m)} \||| \Big\} + \||| A_{u+1,0}^{(m)} - A_{u,j_{u-1}}^{(m)} \||| \Big] \\
(2.3)\quad &\leq \||| \theta_m - \theta \||| \Big[\sum_{u=0}^{h} j_{u+1} U^{q_0 - 1} + \sum_{u=0, q_{u-1} \neq 0}^{h} q_{u+1} U^{q_{u+1}-1} \Big] \\
&\leq \||| \theta_m - \theta \||| \, nU^n.
\end{aligned}$$

Hence

$$\begin{aligned}
&\||| K_\lambda(F_n^m) - K_\lambda(F_n) \||| \\
&\leq \sum_{q_0 + \cdots + q_h = n} n! \frac{| \omega_1 |^{q_1} \cdots | \omega_h |^{q_h}}{q_1! \cdots q_h!} \\
(2.4)\quad & \sum_{j_1 + \cdots + j_{h+1} = q_0} \int_{\triangle_{q_0; j_1, \cdots, j_{h+1}}} \||| \theta_m - \theta \||| \, nU^n \, d\Big(\prod_{p=1}^{h+1} \prod_{i=1}^{j_p} \mu \Big)(s_{p-1,i}) \\
&\leq nU^n \||| \theta_m - \theta \||| | \mu | (a,b)^n.
\end{aligned}$$

Let $\epsilon > 0$ be given. Taking m in $\mathbf{N}$ such that $\||| \theta_m - \theta \||| < \frac{\epsilon}{nU^n |\eta|^n (a,b)}$, we have $\||| K_\lambda(F_n^m) - K_\lambda(F_n) \||| < \epsilon$, as desired. □

Corollary 2.4. *We assume the hypothesis of* Theorem 2.3 *and* $< \theta_n >$ *converges to* θ *uniformly s-a.e.. Then* $\sum_{n=0}^{\infty} \frac{1}{n!} K_\lambda \Big[\Big(\int_a^b \theta_m(s, x(s)) \, d\eta(s) \Big)^n \Big]$ *converges to* $K_\lambda \Big[\exp \Big(\int_a^b \theta(s, x(s)) d\eta(s) \Big) \Big]$ *uniformly.*

Proof. By Theorem 2.3, let M be an upper bound of $\{ \| \theta_m \|_\infty | \, m \in \mathbf{N} \} \cup \{ \| \theta \| \}$ and let $U = \max\{\mathrm{M}, 1\}$

$$\| \sum_{n=0}^{k} \frac{1}{n!} K_\lambda \Big[\Big(\int_a^b \theta_m(s,x(s))d\eta(s) \Big)^n \Big] - \sum_{n=0}^{k} \frac{1}{n!} K_\lambda \Big[\Big(\int_a^b \theta(s,x(s))d\eta(s) \Big)^n \Big] \|$$

$$\begin{aligned} &\leq \sum_{n=1}^{k} \frac{1}{(n-1)!} U^n \mid \eta \mid^n (a,b) \parallel \theta_n - \theta \parallel \\ &\leq U \mid \eta \mid (a,b) \exp(U \mid \eta \mid (a,b)) \parallel \theta_n - \theta \parallel \longrightarrow 0 \end{aligned}$$

as $m \longrightarrow \infty$ for all $k \in \mathbf{N}$.
Hence, by the property of uniform convergence,

$$\begin{aligned} &\lim_{m \longrightarrow \infty} \sum_{n=0}^{\infty} \frac{1}{n!} K_\lambda \Big[\Big(\int_a^b \theta_m(s,x(s))d\eta(s) \Big)^n \Big] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \lim_{m \longrightarrow \infty} K_\lambda \Big[\Big(\int_a^b \theta_m(s,x(s))d\eta(s) \Big)^n \Big] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} K_\lambda \Big[\Big(\int_a^b \theta(s,x(s))d\eta(s) \Big)^n \Big] \\ &= K_\lambda \Big[\exp \Big(\int_a^b \theta(s,x(s))d\eta(s) \Big) \Big], \quad \text{as desired.} \quad \square \end{aligned}$$

We will treat the stability theorem in the measures. Let η and η_m ($m = 1, 2, \cdots$) be in $M(a,b)$ such that η_n converges to η in the total variation norm.

THEOREM 2.5. *Let* $F_n^m(x) = \Big(\int_a^b \theta(s,x(s))d\eta_m(s) \Big)^n$. *Suppose that* $< \eta_m >$ *converges to* η *in the total variation norm. Then* $K_\lambda(F_n^m)$ *converges to* $K_\lambda(F_n)$ *in the operator norm topology.*

Proof. Let

$$T_m = \sup \Big\{ \mid x_1^n - x_2^n \mid \ : \ \mid x_1 - x_2 \mid \leq \parallel \theta \parallel_\infty \mid \eta \mid, \quad \mid x_1 \mid \leq \parallel \theta \parallel_\infty \mid \eta_m \mid \text{ and } \mid x_2 \mid \leq \parallel \theta \parallel_\infty \mid \eta \mid \Big\}. \tag{2.5}$$

Then

$$
\begin{aligned}
&\| F_n^m(x) - F_n(x) \| \\
(2.6) \qquad &= \| \left(\int_a^b \theta(s, x(s))\, d\eta_m(s) \right)^n - \left(\int_a^b \theta(s, x(s))\, d\eta(s) \right)^n \| \le T_m.
\end{aligned}
$$

And

$$
\begin{aligned}
&\| K_\lambda(F_n^m)\psi - K_\lambda(F_n)\psi \|_{p,\infty} \\
&= \sup_{\lambda>0} \Big[\int_{\Omega_\lambda} \mid \int_{C_0(\mathbf{B})} F_n^m(\lambda^{-\frac{1}{2}} y + x)\psi(\lambda^{-\frac{1}{2}} y(b) + x) dm_{\mathbf{B}}(y) \\
&\quad - \int_{C_0(\mathbf{B})} F_n(\lambda^{-\frac{1}{2}} y + x)\psi(\lambda^{-\frac{1}{2}} y(b) + x) dm_{\mathbf{B}}(y) \mid^p dm_\lambda(x) \Big]^{\frac{1}{p}} \\
&= \sup_{\lambda>0} \Big[\int_{\Omega_\lambda} \mid \int_{C_0(\mathbf{B})} \Big(F_n^m(\lambda^{-\frac{1}{2}} y + x) - F_n(\lambda^{-\frac{1}{2}} y(b) + x) \\
&\quad \psi(\lambda^{-\frac{1}{2}} y(b) + x) \Big)\, dm_{\mathbf{B}}(y) \mid^p dm_\lambda(x) \Big]^{\frac{1}{p}} \\
(2.7) \qquad &\le \sup_{\lambda>0} \Big[\int_{\Omega_\lambda} \Big(\int_{C_0(\mathbf{B})} \mid (F_n^m(\lambda^{-\frac{1}{2}} y + x) - F_n(\lambda^{-\frac{1}{2}} y(b) + x)) \\
&\quad \psi(\lambda^{-\frac{1}{2}} y(b) + x) \mid\ dm_{\mathbf{B}}(y) \Big)^p dm_\lambda(x) \Big]^{\frac{1}{p}} \\
&\le \sup_{\lambda>0} T_m \Big[\int_{\Omega_\lambda} \int_{\mathbf{B}} \mid \psi(\lambda^{-\frac{1}{2}} \sqrt{b-a} z + x) \mid^p dm(z) dm_\lambda(x) \Big]^{\frac{1}{p}} \\
&\le \sup_{\lambda>0} T_m \int_{\mathbf{B}} \Big[\int_{\Omega_\lambda} \mid \psi(\lambda^{-\frac{1}{2}} \sqrt{b-a} z + x) \mid^p dm(z) dm_\lambda(x) \Big]^{\frac{1}{p}} \\
&= \| C_{\frac{\lambda}{b-a}} \psi \|_{p,\infty} = T_m \| \psi \|_{p,\infty}.
\end{aligned}
$$

Hence $||| K_\lambda(F_n^m) - K_\lambda(F_n) ||| < T_m$. Now

$$
\begin{aligned}
&\mid \Big(\int_a^b \theta(s, x(s)) d\eta_m(s) \Big)^n - \Big(\int_a^b \theta(s, x(s)) d\eta(s) \Big)^n \mid \\
&\le \Big(\mid \int_a^b \theta(s, x(s)) d\eta_m(s) - \int_a^b \theta(s, x(s)) d\eta(s) \mid \Big)
\end{aligned}
$$

$$\times\Big[\,\Big|\Big(\int_a^b \theta(s,x(s))d\eta_m(s)\Big)^{n-1} + \Big(\int_a^b \theta(s,x(s))d\eta_m(s)\Big)^{n-1}$$
$$\int_a^b \theta(s,x(s))\,d\eta(s) + \cdots + \Big(\int_a^b \theta(s,x(s))d\eta(s)\Big)^{n-1}\Big|\,\Big]$$
$$\leq \Big|\int_a^b \theta(s,x(s))d\eta_m(s) - \int_a^b \theta(s,x(s))d\eta(s)\Big|\,(\|\theta\|_\infty^{n-1} K)$$
$$\leq \|\theta\|_\infty^n \mid \eta_m - \eta \mid K,$$

where $K = \mid \eta_m \mid^{n-1} + \mid \eta_m \mid^{n-2} \mid \eta \mid + \cdots + \mid \eta_m \mid\mid \eta \mid^{n-2} + \mid \eta \mid^{n-1}$. Hence

$$\| K_\lambda(F_n^m)\psi - K_\lambda(F_n)\psi \|_{p,\infty} \leq \|\theta\|_\infty^n \mid \eta_m - \eta \mid \|\psi\|_{p,\infty} K \longrightarrow 0$$

as $m \longrightarrow \infty$ which implies the conclusion. □

From the above results, we have directly following Corollary.

COROLLARY 2.6. *Suppose a bounded sequence* $<\theta_n>$ of Borel *measurable functions on* $\mathbf{B}$ *converges to* θ *uniformly s-a.e. and a sequence* $<\eta_n>$ *of* Borel *measures on* $[a,b]$ *converges to* η *in the total variation norm. For three natural numbers* n, m, l, *and* x *in* $C(\mathbf{B})$, *we let*

$$F_n^{m,l}(x) = \Big(\int_a^b \theta_m(s,x(s))d\eta_l(s)\Big)^n$$

whenever the integral exists.
Then

$$\lim_{m,l\longrightarrow\infty} \sum_{n=0}^{\infty} \frac{1}{n!} K_\lambda(F_n^{m,l}(x)) = K_\lambda\Big[\exp\Big(\int_a^b \theta(s,x(s))d\eta(s)\Big)\Big]$$

in the uniform operator topology.

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