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THE GENERATORS OF COMPLETE INTERSECTION

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ABSTRACT. We classify complete intersections I of grade 3 in a regular local ring $(R, \mathfrak{m})$ by the number of minimal generators of a minimal prime ideal P over I. Here P is either a complete intersection or a Gorenstein ideal which is not a complete intersection.

1. Introduction

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring with a residue field $k = R/\mathfrak{m}$ unless stated otherwise. Let I be a perfect ideal of grade g. The *type* $r(I)$ of I is the dimension of the k-vector space $\operatorname{Ext}_R^g(k, R/I)$, equivalently, if

$$\mathbb{F} : 0 \to F_g \to F_{g-1} \to \cdots \to F_1 \to R \to R/I \to 0$$

is a minimal free resolution of R/I, then $r(I)=$ rank(F_g). An ideal I of grade g is Gorenstein if $r(I) = 1$, a complete intersection if it is minimally generated by g elements, and an almost complete intersection if it is minimally generated by $g+1$ elements.

If $g = 2$ and I is generated by m elements, then I is generated by the $(m-1) \times (m-1)$ minors of an $m \times (m-1)$ matrix [4], i.e., the minimal free resolution of R/I is

$$\mathbb{F} : 0 \to R^{m-1} \xrightarrow{\mathbf{X}} R^m \xrightarrow{\mathbf{x}} R \to R/I \to 0$$

where $\mathbf{X}$ is an $m \times (m-1)$ matrix, and $\mathbf{x} = (x_1, x_2, \cdots, x_m)$, and $x_i = (-1)^{i+1}\Delta_i(\mathbf{X})$, and $\Delta_i(\mathbf{X})$ is the determinant of submatrix of $\mathbf{X}$ formed by omitting the i-th row of $\mathbf{X}$.

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If $g = 3$, then every Gorenstein ideal of grade 3 in a Noetherian local ring is an ideal generated by the maximal order pfaffians of some alternating matrix [3], i.e., the minimal free resolution of R/I is

$$\mathbb{F} : 0 \to R \xrightarrow{\mathbf{x}^T} R^m \xrightarrow{\mathbf{X}} R^m \xrightarrow{\mathbf{x}} R \to R/I \to 0$$

where m is an odd integer, and $\mathbf{X}$ is an $m \times m$ alternating matrix, and $\mathbf{x} = (x_1, x_2, \cdots, x_m)$, and $x_i = (-1)^{i+1}\mathrm{Pf}_i(\mathbf{X})$, and $\mathrm{Pf}_i(\mathbf{X})$ is the pfaffian of an alternating submatrix of $\mathbf{X}$ formed by omitting the i-th row and the corresponding column of $\mathbf{X}$.

If $g = 4$, then the situation is more complicated. In this case, we need to investigate the algebra structure on a free resolution $\mathbb{F}$ of R/I since if we can give the algebra structure on it, then we can induce the algebra structure on $\Lambda_\bullet = \mathrm{Tor}_\bullet^R(R/I, k)$, which agrees with the usual multiplication on the homology algebra. So we mean that Λ_1 is the first homology of $\mathbb{F} \otimes k$. Andrew Kustin and Matthew Miller[7] have founded $\dim_k \Lambda_1^2$ to be an useful invariant in distinguishing resolutions of different form in this case. All known examples fall in one of the following cases.

(1) $\Lambda_1^2 = \Lambda_2$ if and only if I is a complete intersection.
(2) $\dim\Lambda_1^2 = (1/2)\dim\Lambda_2$. This class includes the hypersurface sections of a local Gorenstein ring of codimension three.
(3) $\dim\Lambda_1^2 = 3$. This class includes a family of specializations that are produced from perfect almost complete intersections and their canonical modules.
(4) $\Lambda_1^2 = 0$. This class includes the ideal generated by the $(m-1) \times (m-1)$ minors of an $m \times m$ matrix[5].

However a complete structure theorem for Gorenstein ideals of grade 4 has not founded and we don't know whether the free resolutions of Gorenstein ideals of grade 4 have always the algebra structure or not.

In this paper we do mainly investigate complete intersections of grade 3 in a regular local ring. We adopt the linkage theory, structure theorems for Gorenstein ideals of grade 3, and almost complete intersections of grade 3 to characterize complete intersections of grade 3. In section 2 we review the concept related to alternating matrices and well-known results. In section 3 we provide a full description of the complete intersections of grade 3 in a regular local ring in term of the minimal generators.

2. Alternating matrix and Gorenstein ideal of grade 3

An $m \times m$ matrix $X = (x_{ij})$ is said to be *alternating* if $x_{ji} = -x_{ij}$ and all its diagonal entries are zero. The pfaffian of an alternating matrix X is defined as a square root of its determinant and denoted by $\mathrm{Pf}(X)$. If $s < m$, we let $X(i_1, i_2, \cdots, i_s)$ denote the alternating matrix obtained by deleting rows and columns $i_1, i_2, \cdots, i_s$ of X. Let (i) denote the multi-index $(i_1, i_2, \cdots, i_s)$. Let $\theta(i)$ denote the sign of the permutation that rearranges (i) in increasing order. If (i) has a repeat index, then we set $\theta(i) = 0$. Let $\tau(i)$ be the sum of the entries of (i). Define

$$X_{(i)} = (-1)^{\tau(i)+1} \cdot \theta(i) \cdot \mathrm{Pf}(X(i)). \tag{2.1}$$

If $s = m$, then $X_{(i)} = \pm 1$ and if $s > m$, then $X_{(i)} = 0$.
The $(m-1)$-th order pfaffians of X are defined as the pfaffians of $(m-1) \times (m-1)$ alternating submatrices obtained by deleting a row and the corresponding column of X. It is well-known from the linear algebra that if m is an odd, then the determinant of X is zero and that if m is an even, then the determinant is a square of the pfaffian of X. Let $f : F \to G$ be a map of free R-modules. We define $\mathrm{Pf}_s(f)$ to be the ideal generated by all the s-th order pfaffians of f. D. Buchsbaum and D. Eisenbud gave a structure theorem for the Gorenstein ideals of grade 3.

THEOREM 2.1[3]. *Let $(R, \mathfrak{m})$ be a Noetherian local ring.*

(1) *Let $m \geqslant 3$ be an odd integer. Let F be a free R-module with its rank $F = m$. Let $f : F^* \to F$ be the alternating map whose image is contained in $\mathfrak{m}F$. Suppose that $Pf_{m-1}(f)$ has grade 3. Then $Pf_{m-1}(f)$ is a Gorenstein ideal minimally generated by m elements.*
(2) *Every Gorenstein ideal of grade* 3 *arises in* (1).

Now we review some of the linkage theory.

DEFINITION 2.2[9]. Let I and J be two ideals in a Gorenstein ring R (not necessarily local).

(1) I and J are said to be linked (with respective to α)(we write $I \sim J$) if there exists an R-regular sequence $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_g$ in $I \cap J$ such that $J = (\alpha) : I$ and $I = (\alpha) : J$.

(2) I and J are said to be *geometrically linked* by α if they have no common components and $I \cap J = (\alpha)$.

PROPOSITION 2.3[3]. *Let I and J be perfect ideals of the same grade g in R, and suppose that I is linked to J by an R-regular sequence $x_1, x_2, \cdots, x_g$.*

(1) *If J is Gorenstein, then I is an almost complete intersection.*

(2) *If I is an almost complete intersection and $x_1, x_2, \cdots, x_g$ form part of a minimal set of generators for I, then J is Gorenstein.*

Let I be a perfect ideal of grade g in a Gorenstein local ring R and $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_g$ an R-sequence properly contained in I. Proposition 1.3 in [9] implies that $J = (\alpha) : I$ and I are linked (with respective to α) and J is the perfect ideal of grade g. If M is a finitely generated R-module, then we denote by $\mu(M)$ the number of the minimal generators of M.

COROLLARY 2.4[8]. *Let I be a Gorenstein ideal of grade $g \geqslant 1$ in the Gorenstein local ring R and let K be a complete intersection of grade g which is properly contained in I. Then $K : I$ is a complete intersection if and only if I is a complete intersection and $\mu(I/K) = 1$.*

3. The minimal prime ideal of complete intersection of grade 3

In this section, we investigate minimal prime ideals of complete intersections of grade 3 in terms of minimal generators and describe them by means of pfaffians. Let $(R, \mathfrak{m})$ be a regular local ring with $\dim R = n$. We recall that $x_1, x_2, \cdots, x_i \in \mathfrak{m}$ is a subset of a regular system of parameters of R if and only if the images of $x_1, x_2, \cdots, x_i$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over a field k. M. Auslander and D. Buchsbaum[1] proved that every regular local ring is an unique factorization domain. Now we investigate a class of complete intersections of grade g whose minimal prime ideal is a complete intersection when R is a U.F.D.

THEOREM 3.1. *Let $(R, \mathfrak{m})$ be the regular local ring with dim $R = n$ and let $I = (x_1, x_2, \cdots, x_g)$ be a complete intersection of grade g.*

(1) *If each $x_h = c_{h1}c_{h2}\cdots c_{hm_h}$ where c_{hi} is irreducible in $\mathfrak{m} - \mathfrak{m}^2$ for every $i(1 \leqslant i \leqslant m_h)$, then there exists a minimal prime ideal generated by irreducible factors c_{hi} over I which is a complete intersection.*

(2) *Let x_j be in (1) for every $j(1 \leqslant j \leqslant l)$ with $l < g$ and let others be irreducible elements belonging to $\mathfrak{m}^s$ for some $s > 1$. Let $c_{1^*}, c_{2^*}, \cdots, c_{l^*}$ be a sequence of irreducible elements in R such that grade $(c_{1^*}, c_{2^*}, \cdots, c_{l^*}) = l$. If the image of x_t in $R/(c_{1^*}, c_{2^*}, \cdots, c_{l^*})$ for every $l < t \leqslant g$ is the product of irreducible elements which are all in $\mathfrak{m}/(c_{1^*}, c_{2^*}, \cdots, c_{l^*}) - \{\mathfrak{m}/(c_{1^*}, c_{2^*}, \cdots, c_{l^*})\}^2$, then I has a minimal prime ideal over I which is a complete intersection.*

(3) *If $n = g$, then every minimal prime ideal over I is a complete intersection.*

Proof. (1) Since grade $I = g$, by the assumption, we can choose $c_{1^*}, c_{2^*}, \cdots, c_{g^*}$ in R such that each c_{j^*} is contained in $\mathfrak{m} - \mathfrak{m}^2$ and is an irreducible factor of x_j. Let $P = (c_{1^*}, c_{2^*}, \cdots, c_{g^*})$ be an ideal in R. We will show that P is a minimal prime ideal over I. Clearly, $I \subseteq P$. Since grade $I = g$ and R is local, any proper subset of $\{c_{1^*}, c_{2^*}, \cdots, c_{g^*}\}$ can not generate P. Thus $\{c_{1^*}, c_{2^*}, \cdots, c_{g^*}\}$ is a set of minimal generators of P. Then P has grade g since $I \subseteq P$. Now Nakayama's lemma implies that $c_{1^*} \otimes 1, c_{2^*} \otimes 1, \cdots, c_{g^*} \otimes 1$ forms a basis for the vector space $P \otimes k$. Since $P \subseteq \mathfrak{m}$ and $c_{j^*}$is contained in $\mathfrak{m} - \mathfrak{m}^2$ for every j, $P \otimes k$ is a subspace of $\mathfrak{m} \otimes k = \mathfrak{m}/\mathfrak{m}^2$. Hence the images of $c_{1^*}, c_{2^*}, \cdots, c_{g^*}$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over a field k. Thus $\{c_{1^*}, c_{2^*}, \cdots, c_{g^*}\}$ is a subset of a regular system of parameters of R. We recall that K is an ideal generated by a subset $\{u_1, u_2, \cdots, u_q\}$ of a regular system of parameters of R if and only if R/K is a regular local ring with its dimension $n - q$. So R/P is a regular local ring. Since every regular local ring is an integral domain, P is a prime ideal. It follows from

our construction that P is minimal over I and a compete intersection. Since $I \subseteq P$, P is a minimal prime ideal over I with grade $P = g$.
(2) Since $\{x_i\}_{i=1}^{l}$ satisfies the assumption of (1), we obtain the R-sequence $c_{1^*}, c_{2^*}, \cdots, c_{l^*}$ of irreducible elements in R such that grade $(c_{1^*}, c_{2^*}, \cdots, c_{l^*}) = l$ and $Q = (c_{1^*}, c_{2^*}, \cdots, c_{l^*})$ is a minimal prime ideal over $(x_1, x_2, \cdots, x_l)$. Since every regular local ring is Cohen-Macaulay, grade $(c_{1^*}, c_{2^*}, \cdots, c_{l^*}) = \mathrm{ht}(c_{1^*}, c_{2^*}, \cdots, c_{l^*})$. Hence $\{c_{1^*}, c_{2^*}, \cdots, c_{l^*}\}$ is a subset of system of parameters of R. Hence a ring $\bar{R} = R/(c_{1^*}, c_{2^*}, \cdots, c_{l^*})$ is a regular local ring with maximal ideal $\mathfrak{m}/(c_{1^*}, c_{2^*}, \cdots, c_{l^*})$. Since I has grade g, $\{x_i\}_{i=l+1}^{g}$ is $\bar{R}$-sequence. Let $\bar{x}_i$ be the image of x_i in $\bar{R}$ for every $i(l+1 \leqslant i \leqslant g)$. Since $\{\bar{x}_i\}_{i=l+1}^{g}$ does also satisfy the assumption of (1) under the regular local ring $\bar{R}$, we use the same method as in the proof of (1) to obtain the $\bar{R}$-sequence $\{\bar{c}_{i^*}\}_{i=l+1}^{g}$ of irreducible elements in $\bar{R}$ such that grade $(\bar{c}_{\{l+1\}^*}, \bar{c}_{\{l+2\}^*}, \cdots, \bar{c}_{g^*}) = g - l$ and $\bar{c}_{i^*} = c_{i^*} + (c_{1^*}, c_{2^*}, \cdots, c_{l^*})$ for every $i(l+1 \leqslant i \leqslant g)$ and $c_{i^*} \in R$. By (1), we obtain a minimal prime ideal $\bar{Q}'$ such that $\bar{Q}' = (\bar{c}_{\{l+1\}^*}, \bar{c}_{\{l+2\}^*}, \cdots, \bar{c}_{g^*}) \supset (\bar{x}_{l+1}, \bar{x}_{l+2}, \cdots, \bar{x}_g)$. So we set $P = (c_{1^*}, c_{2^*}, \cdots, c_{g^*})$. Then P is a minimal prime ideal and contains I. This follows from the relations

$$R/P \cong \bar{R}/\bar{Q}' \quad \text{and} \quad \text{grade}\, P = g.$$

(3) Since R is Noetherian, R/I is also Noetherian. Since dim $R/I = 0$ and R/I is Noetherian, R/I is Artinian by Akizuki Theorem. Hence every prime ideal of R/I is a maximal ideal in R/I. Since R is a regular local ring with maximal ideal $\mathfrak{m}$, it is a complete intersection of grade g. Thus every minimal prime ideal over I is a complete intersection. □

We note that if $I = (x_1, x_2, \cdots, x_g)$ is a complete intersection of grade g such that every x_i is as in (1) of Theorem 3.1 and P is any minimal prime ideal over I, then P is a complete intersection of grade g. In general, if R is not a U.F.D., then there exists a minimal prime ideal over a complete intersection which is neither a Gorenstein ideal nor an almost complete intersection.

EXAMPLE 3.2. Let k be a field and let x, y, and z be indeterminates. Let $A = k[[x, y, z]]$ be a regular local ring and $R = k[[x^2, x^3, y^2, y^3, z^2, z^3]]$

a subring of A. Since $x^6 = (x^2)^3 = (x^3)^2$, R is not a U.F.D. but an integral domain. Let $I = (x^2, y^2, z^2)$ be an ideal in R. Since x^2, y^2, z^2 are regular sequence on R, I is a complete intersection of grade 3. The minimal prime ideal over I is $P = (x^2, x^3, y^2, y^3, z^2, z^3)$. Clearly, P is neither a complete intersection nor an almost complete intersection of grade 3. By Theorem 2.1, P is not Gorenstein.

The next example shows that there exists a minimal prime ideal over a complete intersection of grade 3 which is a Gorenstein ideal of grade 3 but not a complete intersection of grade 3.

EXAMPLE 3.3. Let $\mathbb{Q}$ be the field of rational numbers and let x, y, z, w, and u be indeterminates. Let $R = \mathbb{Q}[[x, y, z, w, u]]$ be a regular local ring Let $I = (xu - zw, y^2 - xz, w^2 - yu)$ be an ideal in R. Then I is a complete intersection of grade 3. Let $R' = \mathbb{Q}[[s, t]]$ be a regular local ring with s, t indeterminates. Define a map ϕ as follow

$$\begin{aligned} \phi : R = \mathbb{Q}[[x, y, z, w, u]] &\longrightarrow R' = \mathbb{Q}[[s, t]] \\ 1 &\longmapsto 1 \\ x &\longmapsto s^5 \\ y &\longmapsto s^4 t \\ z &\longmapsto s^3 t^2 \\ w &\longmapsto s^2 t^3 \\ u &\longmapsto t^5. \end{aligned}$$

Clearly, ϕ is a well-defined ring homomorphism. $\operatorname{Im}\phi = \mathbb{Q}[[s^5, s^4t, s^3t^2, s^2t^3, t^5]]$. Since $\operatorname{Im}\phi$ is an integral domain, $\operatorname{Ker}\phi$ is prime in R. We note that $P = \operatorname{Ker}\phi = (w^2 - yu, xu - zw, xw - yz, y^2 - xz, z^2 - yw)$. Consider the following 5×5 alternating matrix

$$f = \begin{pmatrix} 0 & 0 & y & z & x \\ 0 & 0 & -z & -w & -y \\ -y & z & 0 & u & w \\ -z & w & -u & 0 & 0 \\ -x & y & -w & 0 & 0 \end{pmatrix}.$$

Then by the direct computation, $\mathrm{Pf}_4(f) = P$. Since $\operatorname{grade} P = 3$, Theorem 2.1 implies that P is a Gorenstein ideal of grade 3. We can easily see that P is a minimal prime ideal over I.

We note that there exist complete intersections of grade g over which the number of the minimal generators of the minimal prime ideals is not unique, i.e., if I is a complete intersection of grade g and if both P and P' are minimal prime ideals of I, then $\mu(P) \neq \mu(P')$, but if $\dim R = g$ and I is a complete intersection of grade g, then it is unique. The following example illustrates this case.

EXAMPLE 3.4. Let $\mathbb{Q}$ be the field of rational numbers and let x, y, z, w, u, and t be indeterminates. Let $R = \mathbb{Q}[[x, y, z, w, u, t]]$ be a regular local ring over $\mathbb{Q}$. Let $I = (y^2 - xz, tz, wu)$ be an ideal in R. Then I is a complete intersection of grade 3. Clearly, $P = (y, z, w)$ is a minimal prime ideal over I. Let $P' = (y^2 - xz, t, w)$ be an ideal. Let $R' = \mathbb{Q}[[x, y, z, u]]$ be a subring of R. Consider the following isomorphism

$$R/P' \cong R'/(y^2 - xz).$$

Since R' is a regular local ring and $y^2 - xz$ is irreducible in R', $R'/(y^2 - xz)$ is an integral domain and hence P' is prime over I. We can easily check that P' is minimal over I. However $\mu(P/I) = 3$ and $\mu(P'/I) = 2$.

To describe the main results about complete intersections of grade 3 in a regular local ring, we need notations.

DEFINITION 3.5. Let R be a regular local ring with $\dim R = n$. Let I be a complete intersection of grade 3 in R. We denote

$$\mathrm{PCI(I)} = \{P \in \mathrm{Spec}(R) \mid P \text{ is minimal over } I \text{ which is a complete intersection}\}$$

and

$$\mathrm{PGor(I)} = \{P \in \mathrm{Spec}(R) \mid P \text{ is minimal over I which is a Gorenstein but not a complete intersection}\}$$

If I has a minimal prime ideal P which is a complete intersection or which is a Gorenstein ideal but not a complete intersection, then we put

$$\bar{\mu}(I) = \min\{\mu(P/I) | P \in \mathrm{PCI}(I) \text{ or } P \in \mathrm{PGor}(I)\}$$

From the results of E. Kunz's [6] and C. Peskine and L. Szpiro[9], we can derive good information on $\bar{\mu}(I)$.

PROPOSITION 3.6[6,9]. *Let R be a Gorenstein ring and K an ideal of R such that dim R = dim R/K and R/K is a Cohen-Macaulay ring. If $K \neq 0$, we have*

(1) *$Ann(Ann\ K) = K$.*
(2) *$R/(Ann\ K)$ is also a Cohen-Macaulay ring.*
(3) $\mu(\mathrm{Ann}\, K) = r(R/K), \mu(K) = r(R/\mathrm{Ann}\, K)$.

Where $r(I)$ denotes the type of I.

Let $(R, \mathfrak{m})$ be a regular local ring of $\dim R = 3$. Let I be a complete intersection of grade 3 and let P be a minimal prime ideal over I. Let $R' = R/I$ and $K = P/I$. Clearly, R' is a Gorenstein ring and $\dim R' = \dim R'/K$ since $\mathrm{grade}\, P = 3$. $\mu(K)$ plays an important role in our characterization of a class of complete intersections of grade 3. Let T be an $m \times m$ alternating matrix. From (2.1), we have $T_{(i,j)} = (-1)^{i+j+1}\mathrm{Pf}(T(i,j))$ where $T(i,j)$ is the $(m-2) \times (m-2)$ alternating submatrix of T formed by deleting rows and columns i, j of T. Similarly, we have $T_{(i,j,h)} = (-1)^{i+j+h+1}\mathrm{Pf}(T(i,j,h))$ where $T(i,j,h)$ is an $(m-3) \times (m-3)$ alternating submatrix of T formed by deleting rows and columns i, j, h of T.

The following two propositions are the explicit versions of Buchsbaum and Eisenbud's structure theorem for an almost complete intersection of grade 3.

PROPOSITION 3.7[2]. *Let m be an even integer with $m > 4$. Let $(R, \mathfrak{m})$ be a Noetherian local ring. If I is an almost complete intersection of grade 3 with type $m-3$, then there is an $m \times m$ alternating matrix T, with entries in $\mathfrak{m}$, such that $I = (Pf(T), T_{(1,2)}, T_{(1,3)}, T_{(2,3)})$.*

PROPOSITION 3.8[2]. *Let m be an odd integer with $m > 3$. Let $(R, \mathfrak{m})$ be a Noetherian local ring. If I is an almost complete intersection* *of grade 3 with type $m-3$, then there is an $m \times m$ alternating matrix T, with entries in $\mathfrak{m}$, such that $I = (T_{(1)}, T_{(2)}, T_{(3)}, T_{(1,2,3)})$.*

Let I and K be ideals in a regular local ring R and $I \subseteq K$. Let $I : K$ denote their ideal quotient. We recall that $\operatorname{Ann}(K/I) = I : K$. Let I be a complete intersection of grade 3 and let $P \in$ PCI(I) or $P \in$ PGor(I). Since $\operatorname{Ann}(P/I) = I : P$, $\mu(P/I)$ determines the type of an ideal $I : P$ by the part (3) of Proposition 3.6.

The following two theorems describe our main results for the case that a minimal prime ideal over I is a complete intersection and for the case that a minimal prime ideal over I is a Gorenstein ideal but not a complete intersection. First of all we consider the case that $J = I : P$ is a complete intersection for some minimal prime ideal P over I which is a complete intersection.

THEOREM 3.9. *Let R be a regular local ring and I a complete intersection of grade 3.*

(1) *If $\bar{\mu}(I) = 0$, then $J = I : P$ is a prime ideal and a complete intersection for some minimal prime ideal P over I which is a complete intersection, and there exists a 3×3 alternating matrix T such that the ideal generated by the pfaffians of order 2 of T is J.*
(2) *If there is a minimal prime ideal P over I which is a complete intersection and $\mu(P/I) = 1$, then $J = I : P$ is a complete intersection of grade 3 and there exists a 3×3 alternating matrix T such that the ideal generated by the pfaffians of order 2 of T is J.*

Proof. (1) Assume that $\bar{\mu}(I) = 0$. Then there exists a minimal prime ideal P over I such that $\mu(P/I) = 0$. So we have $P = I$. Since P is prime, I is prime. In this case, we have $J = I : P = I$. Hence J is a complete intersection of grade 3. By Theorem 2.1, there exists a 3×3 alternating matrix T such that J is the ideal generated by the pfaffians of order 2 of T.
(2) Let P be a minimal prime ideal over I such that P is a complete intersection and $\mu(P/I) = 1$. Let $J = I : P$. We note that $\operatorname{Ann}(P/I) = \{x \in R | x(P/I) = 0\} = I : P = J$. Since $\mu(P/I) = 1$, I is properly

contained in P. Since P is a complete intersection and $\mu(P/I) = 1$, $J = I : P$ is a complete intersection by Corollary 2.4. Since every complete intersection is Gorenstein, Theorem 2.1 gives us that there exists a 3×3 alternating matrix T such that the ideal generated by the pfaffians of order 2 of T is J. $\square$

Next we consider the case that $J = I : P$ is an almost complete intersection for some minimal prime ideal P over I which is a complete intersection or which is a Gorenstein ideal but not a complete intersection.

THEOREM 3.10. *Let $(R, \mathfrak{m})$ be a regular local ring and let I be a complete intersection of grade 3.*

(1) *If $\bar{\mu}(I)$ is even and greater than 1, then there exists an odd integer m, and an $m \times m$ alternating matrix T, with entries in $\mathfrak{m}$, such that $J = (T_{(1)}, T_{(2)}, T_{(3)}, T_{(1,2,3)})$.*
(2) *If $\bar{\mu}(I)$ is odd and greater than 1, then there exists an even integer m, and an $m \times m$ alternating matrix T, with entries in $\mathfrak{m}$, such that $J = (Pf(T), T_{(1,2)}, T_{(1,3)}, T_{(2,3)})$.*

Proof. (1) Assume that $\bar{\mu}(I)$ is even and greater than 1. Then there exists a minimal prime ideal P over I such that $\mu(P/I) > 1$ is even and P is a complete intersection or a Gorenstein ideal but not a complete intersection. Let $J = I : P$. Since $\mu(P/I) > 1$, J is not a complete intersection by Corollary 2.4. Since every complete intersection is Gorenstein, P is a Gorenstein ideal. By Proposition 2.3, J is an almost complete intersection. Let $t = \mu(P/I)$. Then t is even. We note that $\mathrm{Ann}(P/I) = I : P = J$. By the part (3) of Proposition 3.6, t is the type of J. Let $m = t + 3$. Then m is an odd integer with $m > 3$. By Proposition 3.8, there exists an $m \times m$ alternating matrix T such that $J = (T_{(1)}, T_{(2)}, T_{(3)}, T_{(1,2,3)})$.
(2) The proof of the second part is essentially the same as that of (1). $\square$

We illustrate the cases of Theorem 3.10.

EXAMPLE 3.11. Let $\mathbb{Q}$ be the field of rational numbers and let x, y, z, w, t, and u be indeterminates. Let $R = \mathbb{Q}[[x, y, z, w, t, u]]$ be a regular local ring with maximal ideal $\mathfrak{m} = (x, y, z, w, t, u)$.

(1) Let $I = (x^2, y^2 - xz, w^2)$ be an ideal in R. Then I is a complete intersection of grade 3. The minimal prime ideal over I is (x, y, w). Hence $\bar{\mu}(I) = 3$.

(2) Let $I = (x^2 + wt, y^2 - z^2, u^2)$ be an ideal in R. Then I is a complete intersection of grade 3. The minimal prime ideals over I are $(x^2 + wt, y - z, u)$ and $(x^2 + wt, y + z, u)$. Hence $\bar{\mu}(I) = 2$.

Let $R = k[[x_1, x_2, \cdots, x_g]]$ be a regular local ring with dim $R = g$ over a field k. Let $\mathfrak{m}$ be the maximal ideal of R and let I be an ideal of grade g. Since dim $R/I = 0$, every prime ideal over I is maximal and so it is $\mathfrak{m}$. Thus in this case we have the following two propositions.

PROPOSITION 3.12. *If $(R, \mathfrak{m})$ is a regular local ring with dim $R = g$. If I is a complete intersection of grade g, then $\bar{\mu}(I) = \mu(\mathfrak{m}/I)$.*

PROPOSITION 3.13. *If $(R, \mathfrak{m})$ is a regular local ring with dim $R = g$. Let $I = (x_1, x_2, \cdots, x_g)$ be a complete intersection of grade g. If every x_j is in $\mathfrak{m}^s$ for any positive integer $s > 1$ and $1 \leqslant j \leqslant g$, then $\bar{\mu}(I) = g$. In particular, if $g = 3$, then an almost complete intersection $J = I : \mathfrak{m}$ is generated by pfaffians of alternating submatrices of a 6×6 alternating matrix with entries in $\mathfrak{m}$.*

It is worth to notice that the above two theorems are closely related with the minimal free resolutions by mean of the type of $I : P$ where P is a minimal prime ideal over I that is a complete intersection or that is a Gorenstein ideal which is not a complete intersection. For example, if $\bar{\mu}(I) = 1$, then there exists a minimal prime ideal P over I which is a complete intersection or which is a Gorenstein ideal but not a complete intersection such that if

$$\mathbb{F} : 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0$$

is a minimal free resolution of $R/(I : P)$ then the rank $F_n = 1$. On the other hand, if $h = \bar{\mu}(I)$ is even and greater than 1, then there exists a minimal prime ideal P over I which is a complete intersection or which is a Gorenstein ideal but not a complete intersection such that if

$$\mathbb{F} : 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0$$

is a minimal free resolution of $R/(I : P)$ then the rank $F_n = h$.

References

[1] M. Auslander and D. Buchsbaum, *Unique factorization in the regular local ring*, Proc. Natl. Acad. Sci. U.S.A. **45** (1959), 733–734.
[2] A. Brown, *A structure theorem for a Class of Grade Three Perfect Ideals*, J. Algebra **105** (1987), 308–327.
[3] David Buchsbaum and David Eisenbud, *Algebra Structures for finite free resolutions and some Structure Theorems for ideals for codimension* 3, Amer. J. Math. **99 No 3** (1977), 447–485.
[4] L. Burch, *On the ideals of finite homological dimension in local ring*, Proc. Cambridge Philos. Soc. **64** (1968), 941–966.
[5] J. Herzog, *Certain complexes associated to a sequence and matrix*, Manuscripta Math. **12** (1974), 217–248.
[6] Ernst Kunz, *Almost complete intersections are not Gorenstein Rings*, J. Algebra **28** (1974), 111–115.
[7] A. Kustin and M. Miller, *Structure theory for a class of grade four Gorenstein ideals*, Trans. Amer. Math. Soc. **270** (1982), 287–307.
[8] Andrew R. Kustin and Matthew Miller, *Tight Double Linkage of Gorenstein Algebras*, J. Algebra **95** (1985), 384–397.
[9] C. Peskine and L. Szpiro, *Liaison des variétés algébriques*, Invent. Math. **26** (1974), 271–302.

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