Bull. Korean Math. Soc. **37** (2000), No. 4, pp. 843–854

NÉRON SYMBOL ON k-HOLOMORPHIC TORUS

KYUNG AH SHIM AND SUNG SIK WOO

ABSTRACT. S. Turner has shown that a Néron symbol can be calculated from the values of K-meromorphic theta functions corresponding to divisors on K-holomorphic torus of strongly diagonal type. Using an isogeny to a K-holomorphic torus of strongly diagonal type, he constructed a Néron symbol on K-holomorphic torus of diagonal type. In this work, we provide a simple formula of the Néron symbol on the Tate curve. And then we construct the Néron symbol on K-holomorphic torus of diagonal or st rongly diagonal type without using isogenies.

Introduction

Let K be a local field with a nonarchimedean valuation v and A be an abelian variety defined over K. Let $\mathfrak{a}$ be a zero cycle on A of degree 0 such that each component of $\mathfrak{a}$ is K-rational and Δ be a K-rational divisor on A such that $|\Delta| \cap |\mathfrak{a}| = \varnothing$. The theory of distributions developed by Weil and refined by Néron leads to the construction of a bilinear symbol $< \Delta, \mathfrak{a} >_v$ with values in $\mathbb{R}$. The results from Néron's theory which will be referred to in this paper are summerized in section 1.

In section 2, we provide a simple formula of the Néron symbol on the Tate curve.

Beginning in section 3, we recall that the analytic representation of an abelian variety T over K as a nonarchimedean holomorphic torus allows an explicit description of the divisors of T by theta functions. S. Turner [6] has shown that a Néron symbol can be calculated from the values of K-meromorphic theta functions associate to divisors on

Received June 14, 1999.
2000 Mathematics Subject Classification: 14K15, 14G20, 14K25.
Key words and phrases: Néron symbol, K-holomorphic torus of strongly diagonal type, diagonal type, theta functions.

K-holomorphic torus of strongly diagonal type. Using an isogeny to a K-holomorphic torus of strongly diagonal type, he constructed a Néron symbol on K-holomorphic torus of diagonal type. We will construct the Néron symbol on K-holomorphic torus of diagonal or strongly diagonal type without using isogenies.

1. Néron Symbol on Abelian Varieties

Let K be a local field with a nonarchimedean valuation v and A be an abelian variety defined over K and let $v = -\log|\ |_v$. We fix the notations;

$A(K)$	the group of K-rational points of A
$D(A)_K$	the group of K-rational divisors on A
$D_l(A)_K$	the subgroup of $D(A)_K$ of K-rational divisors which are linearly equivalent to 0
$D_a(A)_K$	the subgroup of $D(A)_K$ of K-rational divisors which are algebraically equivalent to 0
$Z_0(A)_K$	the group of zero cycles of degree 0 which components are K-rational.

If $\Delta = (f)$ is principal, for any $\mathfrak{a} \in Z_0(A)_K$ such that $|\Delta| \cap |\mathfrak{a}| = \varnothing$ we let

$$f(\mathfrak{a}) = \prod_{i=1}^{r} f(a_i)^{n_i}, \text{ for } \mathfrak{a} = \sum_{i=1}^{r} n_i(a_i) \in Z_0(A)_K.$$

This value depends only on Δ since the constant disappears when taking the product over the points of a zero cycle of degree 0.

Néron [3] proved the following theorem.

THEOREM 1.1. *Let A be an abelian variety over a field K with an absolute value v. Then there exists a unique pairing*

$$<\ ,\ >_v : \{(\Delta, \mathfrak{a}) \in D_a(A)_K \times Z_0(A)_K; |\Delta| \cap |\mathfrak{a}| = \varnothing\} \to \mathbb{R}$$

satisfying the following properties:

(1) *It is bilinear.*

(2) *If* $\Delta = (f)$ *is principal then* $< (f), \mathfrak{a} >_v = v(f(\mathfrak{a}))$.

(3) *It is invariant under translation i.e.,* $< \Delta_a, \mathfrak{a}_a >_v = < \Delta, \mathfrak{a} >_v$ *for* $a \in A(K)$.

(4) *Let* $a_0 \in A(K)$ *but* $a_0 \notin |\Delta|$. *Then the map* $a \mapsto < \Delta, (a) - (a_0) >_v$ *is bounded on each* v*-bounded subset of* $A(K) \setminus |\Delta|(K)$.

Such a pairing satisfying the conditions of Theorem 1.1 will be called a *Néron symbol.*

2. Néron Symbol on Tate Curves

Let E_K be the Tate curve defined over K. Then there exists a unique period $q \in K^*$ such that $v(q) \neq 0$. We have the analytic parameterization

$$0 \longrightarrow q^{\mathbb{Z}} \longrightarrow K^* \xrightarrow{\pi} E(K) \longrightarrow 0.$$

Let Δ be a divisor on E_K of degree 0. We now recall the definition and some properties of nonarchimedean theta functions on K^*; all is well explained in [1] and [4].

DEFINITION 2.1. A theta function $\theta_\Delta(x)$ of type (Δ, α) on a Tate curve E_K is K-meromorphic function on K^* with $\theta_\Delta(qx) = \alpha\theta_\Delta(x)$ and $\operatorname{div}(\theta_\Delta) = \Delta$ where $\alpha \in K^*$ is determined by Δ up to a power of q.

LEMMA 2.2. *With the notations above we have;*

(1) *There exist a theta function* $\theta_\Delta(x)$ *of type* (Δ, α) *and any two of them only differ by a constant in* K^*.

(2) *If* $\theta_\Delta(x)$ *is a theta function of type* (Δ, α) *then* $x^m\theta_\Delta(x)$ *is a theta function of type* $(\Delta, \alpha q^m)$.

(3) For $x_1, \cdots, x_r \in K^*$ and $m, m_1, \cdots, m_r \in \mathbb{Z}$ with $\sum_{i=1}^r m_i = 0$. *Put* $\Delta = \sum_{i=1}^r m_i \pi(x_i)$ *and* $\alpha = q^m \prod_{i=1}^r x_i^{m_i}$ *then*

$$\theta_\Delta(x) = x^m \prod_{i=1}^r \theta_0(\frac{x}{x_i})^{m_i}$$

with $\theta_0(x) = \prod_{n\geqq 0}(1-q^n x)\prod_{n\geqq 1}(1-q^n x^{-1})$ *is a theta function of type* (Δ, α).

Let θ_Δ be a theta function of type (Δ, α) on K^*. We define

$$\delta_\Delta(x) = v(\theta_\Delta(x)) - \frac{v(\alpha)}{v(q)}v(x), \quad x \in K^* \setminus \pi^{-1}(|\Delta|).$$

It makes sense since $v(q) = -\log|q|_v \neq 0$, $|q|_v < 1$. One immediately checks that $\delta_\Delta(x)$ is q-periodic and therefore induces a map

$$\delta_\Delta : \ \{\mathfrak{a} \in Z_0(E)_K; |\mathfrak{a}| \cap |\Delta| = \varnothing\} \to \mathbb{R}$$

which depends only on Δ (and not on the choice of α and θ_Δ).

THEOREM 2.3. *Let E_K be a Tate curve defined over a local field K. Then the pairing*

$$<\ ,\ >_v: \{(\Delta, \mathfrak{a}) \in D_a(E)_K \times Z_0(E)_K; |\Delta| \cap |\mathfrak{a}| = \varnothing\} \to \mathbb{R}$$

defined by $<\Delta, \mathfrak{a}>_v = \delta_\Delta(\mathfrak{a}')$ *is a Néron symbol over* E_K, *where* $\mathfrak{a} = \sum_{i=1}^k m_i(a_i)$, $\mathfrak{a}' = \sum_{i=1}^k m_i(a_i')$ *and* a_i' *is any preimage of* a_i *under* π.

Proof. We have to check the properties of Theorem 1.1.

(1) It is additive with respect to the second factor by definition. Since $\theta_{\Delta_1+\Delta_2} = \theta_{\Delta_1}\cdot\theta_{\Delta_2}$ up to constants and $v(\alpha_1\cdot\alpha_2) = v(\alpha_1) + v(\alpha_2)$, it is additive with respect to the divisors.

(2) If $\Delta = (f)$ is principal then $\theta_\Delta = f\circ\pi$ up to constants. We get $v(\alpha) = 0$ since $\Delta = \sum_{i=1}^r n_i(x_i)$ is principal if and only if $\sum_{i=1}^r n_i = 0$, and $\alpha = \prod_{i=1}^r x_i^{n_i} = 1$. Therefore we have

$$\delta_\Delta(\mathfrak{a}') = v(\theta_\Delta(\mathfrak{a}')) - \frac{v(\alpha)}{v(q)}v(\mathfrak{a}') = v(\theta_\Delta(\mathfrak{a}')) = v(f\circ\pi(\mathfrak{a}')) = v(f(\mathfrak{a})).$$

(3) Fix some $a' \in K^*$ such that $\pi(a') = a$ where $\pi : K^* \to E_K \cong K^*/(q^{\mathbb{Z}})$. Let Δ be a divisor of algebraically equivalent to zero on E_K

i.e., $\Delta = \sum_{i=1}^{r} m_i \pi(x_i)$ such that $\sum_{i=1}^{r} m_i = 0$ then theta functions of type (Δ, α) and (Δ_a, α') are

$$\theta_\Delta(x) = \prod_{i=1}^{r} \theta_0(\frac{x}{x_i})^{m_i}$$

$$\theta_{\Delta_a}(x) = \prod_{i=1}^{r} \theta_0(\frac{x}{a'x_i})^{m_i}$$

where $\alpha = \prod_{i=1}^{r} x_i^{m_i}$, $\alpha_{a'} = a'^{\sum m_i} \prod_{i=1}^{r} x_i^{m_i} = \alpha$. Thus $\theta_{\Delta_a}(x) = \theta_\Delta(x/a')$. We have

$$\begin{aligned}\delta_{\Delta_a}(a'x) &= v(\theta_{\Delta_a}(a'x)) - \frac{v(\alpha_{a'})}{v(q)} v(a'x) \\ &= v(c \cdot \theta_\Delta(x)) - \frac{v(\alpha)}{v(q)}(v(a') + v(x)) \\ &= v(\theta_\Delta(x)) - \frac{v(\alpha)}{v(q)} v(x) + const.\end{aligned}$$

Therefore we conclude that this pairing is invariant under translation.

(4) The v-boundedness is obvious from the definition of δ_Δ. □

3. Néron Symbol on K-holomorphic Torus

In this section we construct a Néron symbol on the K-holomorphic torus of higher dimension. We recall some basic facts about K-holomorphic torus. Let K be a complete field with respect to a nontrivial nonarchimedean valuation v. The K-affine algebraic group Spec$K[z_1, z_1^{-1}, \cdots, z_n, z_n^{-1}]$ where $z_1, \cdots, z_n$ are algebraically independent over K, induces a K-holomorphic group which we denote G. We will refer this as the K-algebraic torus of dimension n. The charac ter group $\mathrm{Hom}(G, \mathbb{G}_m)$ of G will be denoted by H. It is a subgroup of the multiplicative group A^* of nowhere vanishing K-holomorphic functions of G and consists of all functions $z_1^{\nu_1} \cdots z_n^{\nu_n}$, $\nu_i \in \mathbb{Z}$. Thus H is a free abelian group of rank n. One can prove that every K-holomorphic function $f(x)$

which vanishes nowhere on G is of the form $f(x) = c \cdot z_1^{\nu_1} \cdots z_n^{\nu_n}, c \in K^*$. Thus A^* is the direct product of K^* and the charact er group H.

Let L be a field containing K and let $l_1, \cdots, l_r \in L^*$. Then the L-valued point in G defined by the K-algebra homomorphism $K[z_1, z_1^{-1}, \cdots, z_r, z_r^{-1}] \to L$ which maps z_i to l_i will be denoted by $(l_1, \cdots, l_r)$. In this way the group $G(L)$ of L-valued points of G is canonically identified with $(L^*)^r$ and the z_i may be thought of as a system of coordinates in G. Let Γ be a free discrete subgroup of G of rank r and T the K-holomorphic space G/Γ. We call T a *K-holomorphic torus* following [1]. The group $T(L)$ of L-valued points of T is canonically identified with $(L^*)^r/\Gamma$.

Let $\mathcal{H}_G^*$ be the sheaf of nowhere vanishing K-holomorphic function germs on G. Gerritzen [1] proved;

THEOREM 3.1. *If G is a K-algebraic torus then $H^1(G, \mathcal{H}_G^*)$ is trivial.*

Thus all the divisors on G are principal divisors of meromorphic functions on G. Let Δ be a divisor on the K-holomorphic torus $T = G/\Gamma$ and let $\pi : G \to T$. Then we have a group homomorphism

$$\pi^* : D(T) \to D(G).$$

Since $H^1(G, \mathcal{H}_G^*) = 1$, any divisor on G is induced by a global meromorphic function h on G. If $h(x)$ is a meromorphic function on G which induces the divisor $\pi^*(\Delta)$ then the meromorphic function $h(\gamma x)$ induces the same divisor for any $\gamma \in \Gamma$. Let

$$\zeta_\gamma(x) = \frac{h(\gamma x)}{h(x)}.$$

Then $\zeta_\gamma(x)$ must be a nowhere vanishing holomorphic function on G and we have the equation;

$$\zeta_{\gamma\gamma'}(x) = \zeta_{\gamma'}(\gamma x) \cdot \zeta_\gamma(x).$$

Thus the mapping $\gamma \mapsto \zeta_\gamma(x)$ defines 1-cocycle $\zeta \in \mathcal{Z}^1(\Gamma, A^*)$. And we have an monomorphism

$$\eta : \frac{D(T)}{D_l(T)} \to H^1(\Gamma, A^*).$$

In order to obtain more information about the divisors on the K-holomorphic torus $T = G/\Gamma$, we look at the cohomology group H^1 (Γ, A^*) more closely. We have a canonical exact sequence [1], §3,

$$H \xrightarrow{\ \delta\ } \mathrm{Hom}(\Gamma, K^*) \xrightarrow{\ \alpha\ } H^1(\Gamma, A^*) \xrightarrow{\ \beta\ } \mathrm{Hom}(\Gamma, H).$$

Let $N(T)$ be the image of β in $\mathrm{Hom}(\Gamma, H)$. We denote by $\mathrm{Pic}(T)$ the kernel of the homomorphism β. Gerritzen [1] proved;

LEMMA 3.2. *$H^1(\Gamma, A^*) = Pic\ (T) \times N(T)$.*

Thus any cohomology class $j \in H^1(\Gamma, A^*)$ can be written as a pair $j = (c, \sigma)$, $c \in \mathrm{Pic}(T)$, $\sigma \in N(T)$. Further Gerritzen showed that for any 1-cocycle $\zeta \in Z^1(\Gamma, A^*)$ there is a decomposition

$$\zeta_\gamma = c_\gamma \cdot \sigma(\gamma)$$

where $\gamma \mapsto c_\gamma$ is a homomorphism $c : \Gamma \to K^*$ and $\sigma \in N(T)$. We call ζ *positive* (resp. *positive nondegenerate*) if $|q(\gamma, \sigma(\gamma)| < 1$ whenever $\sigma(\gamma) \neq 1$ (resp. σ is positive and monomorphism). A K-meromorphic function $f(x)$ on G is called a *theta function of type* ζ if $f(x) = \zeta_\gamma(x) \cdot f(\gamma x)$ for any $\gamma \in \Gamma$. Projectivity of T is equivalent to the existence of a certain cohomology class.

THEOREM 3.3 [1]. *A holomorphic torus $T = G/\Gamma$ is projective algebraic if and only if there exists a positive, nondegenerate cohomology class in $H^1(\Gamma, A^*)$.*

Thus by the nonarchimedean GAGA [2], if T is K-holomorphic torus which is projective algebraic then the holomorphic group structure on T is indeed algebraic group structure. This means that T is an abelian variety over K.

THEOREM 3.4 [1]. *Assume that T is K-holomorphic torus which is an abelian variety. Then $j \in H^1(\Gamma, A^*)$ corresponds to an algebraically equivalent to zero divisor class if and only if $j \in Pic\ (T)$.*

Thus we have an isomorphism

$$\frac{\mathrm{Hom}(\Gamma, K^*)}{\delta(H)} \cong \mathrm{Pic}(T).$$

DEFINITION 3.5. An abelian variety defined over K which is representable by a K-holomorphic torus $T = G/\Gamma$ is said to be of strongly diagonal type if there exists a period matrix (γ_{ij}), $1 \leq i, j \leq n$ on T such that $v(\gamma_{ii}) = 1$ and $v(\gamma_{ij}) = 0$ for all $i \neq j$.

DEFINITION 3.6. An abelian variety defined over K which is representable by a K-holomorphic torus $T = G/\Gamma$ is said to be of diagonal type if there exists a symmetric period matrix (γ_{ij}), $1 \leq i, j \leq n$ on T such that $v(\gamma_{ii}) > 0$ and $v(\gamma_{ij}) = 0$ for all $i \neq j$.

Abelian varieties of diagonal type are discussed in [5], II. Note that abelian varieties of strongly diagonal type are not necessarily of diagonal type.

Let T be a K-holomorphic torus of diagonal type or strongly diagonal type which is an abelian variety. Let Δ be a divisor of algebraically equivalent to zero on T. Then there exists a theta function θ_Δ type of ζ such that $\theta_\Delta(\gamma x) = \zeta_\gamma(x) \cdot \theta_\Delta(x)$, by Theorem 3.4, $\zeta_\gamma(x) = c_\gamma$ where $\gamma \mapsto c_\gamma$ is a homomorphism $c_{\theta_\Delta} : \Gamma \to K^*$. Let $< z_1, \cdots, z_n >$ be a basis of the character group H and $< \gamma_1, \cdots, \gamma_n >$ a basis of Γ. Define a bilinear map $q : \Gamma \times H \to K^*$ by $q(\gamma_i, z_j) = \gamma_{ij}$. Extend this to a bilinear form $q : G \times H \to K^*$. Define

$$\delta_\Delta(x) \equiv v(\theta_\Delta(x)) - \sum_{j=1}^{n} \frac{v(c_{\theta_\Delta}(\gamma_j))}{v(\gamma_{jj})} v(q(x, z_j)), \ x \in (K^*)^n \setminus \pi^{-1}(|\Delta|),$$

where $c_{\theta_\Delta} : \Gamma \to K^*$ is the cohomology class corresponding to θ_Δ. Since $v(\gamma_{jj}) \neq 0$, for $j = 1, \cdots, n$, this definition makes sense.

LEMMA 3.7. *With the notation above we have;*

(1) δ_Δ *is* Γ*-periodic.*
(2) δ_Δ *is determined up to a constant depending on the choice of the theta functions.*

Proof. (1) δ_Δ is Γ-periodic. For any $\gamma_i,\ i=1,\cdots,n$,

$$\begin{aligned}\delta_\Delta(\gamma_i x) &= v(\theta_\Delta(\gamma_i x)) - \sum_{j=1}^{n}\frac{v(c_{\theta_\Delta}(\gamma_j))}{v(\gamma_{jj})}v(q(\gamma_i x, z_j))\\ &= v(c_{\theta_\Delta}(\gamma_i)) - \sum_{j=1}^{n}\frac{v(c_{\theta_\Delta}(\gamma_j))}{v(\gamma_{jj})}v(q(\gamma_i, z_j))\\ &\quad + v(\theta_\Delta(x)) - \sum_{j=1}^{n}\frac{v(c_{\theta_\Delta}(\gamma_j))}{v(\gamma_{jj})}v(q(x, z_j))\\ &= v(c_{\theta_\Delta}(\gamma_i)) - v(c_{\theta_\Delta}(\gamma_i)) + \delta_\Delta(x) = \delta_\Delta(x)\end{aligned}$$

since $v(\gamma_{ij})=0,\ v(q(\gamma_i,z_j))=0$ if $i\neq j$.

(2) Let θ'_Δ be another theta function associated to Δ then $\theta'_\Delta = c\cdot\chi\cdot\theta_\Delta$ for some $c\in K^*$ and $\chi\in H$. Then $c_{\theta'_\Delta}(\gamma)=\chi(\gamma)\cdot c_{\theta_\Delta}(\gamma)$ for any $\gamma\in\Gamma$. Now we may assume $c=1$ because multiplying c changes the value of δ_Δ by a constant. We compute

$$\begin{aligned}\delta_\Delta(x) &= v(\theta'_\Delta(x)) - \sum_{j=1}^{n}\frac{v(\chi(\gamma_j)\cdot c_{\theta_\Delta}(\gamma_j))}{v(\gamma_{jj})}v(q(x, z_j))\\ &= v(\chi(x)\cdot\theta_\Delta(x)) - \sum_{j=1}^{n}\frac{v(\chi(\gamma_j)\cdot c_{\theta_\Delta}(\gamma_j)}{v(\gamma_{jj})}v(q(x, z_j))\\ &= v(\chi(x)) + \delta_\Delta(x) - \sum_{j=1}^{n}\frac{v(\chi(\gamma_j))}{v(\gamma_{jj})}v(q(x, z_j)).\end{aligned}$$

But if $\chi=z_i$ then $z_i(\gamma_j)=0$ for $i\neq j$ (T is a diagonal type or strongly diagonal type). Hence

$$v(\chi(x)) = \sum_{j=1}^{n} v(\chi(\gamma_j))\frac{v(q(x,z_j))}{v(\gamma_{jj})}.$$

Thus δ_Δ is independent of the choice of theta functions up to a constant. $\square$

By extending δ_Δ on the group of zero cycles of degree 0, we have a well determined map

$$\delta_\Delta : \{\mathfrak{a} \in Z_0(T)_K ; |\mathfrak{a}| \cap |\Delta| = \varnothing\} \to \mathbb{R}.$$

The following theorem can be deduced from a formula in Werner [7], but we provide a simple proof;

THEOREM 3.8. *Let T be a K-holomorphic torus of diagonal or strongly diagonal type. Then the pairing between the group of divisors of algebraically equivalent to zero and the group of zero cycles of degree 0*

$$< \, , \, >_v : \{(\Delta, \mathfrak{a}) \in D_a(T)_K \times Z_0(T)_K ; |\Delta| \cap |\mathfrak{a}| = \varnothing\} \to \mathbb{R}$$

defined by $< \Delta, \mathfrak{a} >_v = \delta_\Delta(\mathfrak{a}')$ *is a Néron symbol over T, where* $\mathfrak{a} = \sum_{i=1}^k m_i(a_i)$, $\mathfrak{a}' = \sum_{i=1}^k m_i(a_i')$ *and a_i' is any preimage of a_i under* π.

Proof. We need to check the properties of Theorem 1.1.

(1) Additivity in Δ follows from the fact that we can choose a theta function for $\Delta_1 + \Delta_2$ as $\theta_{\Delta_1}\theta_{\Delta_2}$ and the equality $c_{\theta_{\Delta_1}\theta_{\Delta_2}} = c_{\theta_{\Delta_1}} c_{\theta_{\Delta_1}}$.

(2) If $\Delta = (f)$ is principal, then we can choose $\theta_\Delta = f \circ \pi$ as a theta function. Since

$$\frac{f \circ \pi(\gamma_i x)}{f \circ \pi(x)} = 1$$

i.e., $c_{\theta_\Delta}(\gamma_i) = 1$, we have $< \Delta, \mathfrak{a} >_v = \delta_\Delta(\mathfrak{a}') = v(\theta_\Delta(\mathfrak{a}')) = v(f(\mathfrak{a}))$.

(3) Fix some $a' \in (K^*)^n$ such that $\pi(a') = a$ where $\pi : (K^*)^n \to T$. Let Δ, Δ_a be divisors of algebraically equivalent to zero on T. Choose a theta function θ_Δ for Δ and $\theta_\Delta \circ \tau_{-a'}$ for Δ_a where $\tau_{-a'}$ is a translation by $-a'$. Then $c_{\theta_\Delta} = c_{\theta_{\Delta_a}}$. Thus we have

$$\begin{aligned}
\delta_{\Delta_a}(a'x) &= v(\theta_{\Delta_a}(a'x)) - \sum_{j=1}^{n} \frac{v(c_{\theta_{\Delta_a}}(\gamma_j))}{v(\gamma_{jj})} v(q(a'x, z_j)) \\
&= v(c \cdot \theta_\Delta(x)) - \sum_{j=1}^{n} \frac{v(c_{\theta_\Delta}(\gamma_j))}{v(\gamma_{jj})} \{v(q(x, z_j)) - v(q(a', z_j))\} \\
&= \delta_\Delta(x) + const
\end{aligned}$$

where $c \in K^*$. Consequently, this pairing is invariant under translation.
(4) Boundedness is clear from our definition. $\square$

Let T be an arbitrary K-holomorphic torus which is an abelian variety defined over K and Δ a divisor of algebraically equivalent to zero. Then, by Theorem 3.4, the cohomology class corresponding to a divisor of algebraically equivalent to zero is given by a homomorphism $c_\Delta : \Gamma \to K^*$. If the values of c_Δ are in the group of units U of K then the expression for the Néron symbol is particularly simple.

THEOREM 3.9. *If Δ is a divisor of algebraically equivalent to 0 and if the corresponding cohomology class c_Δ takes values in the group of units U then we have*

$$< \Delta, \mathfrak{a} >_v = v(\theta_\Delta(\mathfrak{a})).$$

Proof. Since v is trivial on the group of units, the second term of δ_Δ in Theorem 3.8, disappears. Thus this result follows from Theorem 3.8. $\square$

Now we will show that for most of divisors the corresponding cohomology class c_Δ takes values in the group of units. First we prove a lemma which may be well known.

LEMMA 3.10. *Let $K/\mathbb{Q}_p$ be a finite extension. Let U be the group of units in K. Then $U^n \subseteq (\bar{K}^*)^n$ is* Zariski *dense where $\bar{K}$ denotes the algebraic closure of K.*

Proof. We induct on n. Suppose $n = 1$, then we have to show that $U \subseteq \bar{K}^*$ is Zariski dense. We need to show that every basic open set intersects U. But every basic open of $\bar{K}^*$ is obtained by removing a finite set of points. Since U is infinite, we see that every basic open set intersects U. Now suppose $n > 1$, let $f \in \bar{K}[x_1, \cdots, x_n]$. We need to show that $(\bar{K}^*)^n \smallsetminus V(f)$ intersects $U \times \cdots \times U (= U^n)$. Obviously there exists a Zariski open set $V \subseteq K^*$ such that whenever $t \in V$ the set $V(f(t, x_1, \cdots, x_n))^c$ is a Zariski open set. Since V is a Zariski open set we have that $U \cap V \neq \varnothing$. Hence there is a $u \in U$ such that $V(f(u, x_2, \cdots, x_n))^c \cap U^{n-1} \neq \varnothing$. $\square$

The group $\mathrm{Hom}(\Gamma, K^*)$ can be identified with $(K^*)^n$ where $n = rank$ (Γ). As in [1], we denote the group $\delta(H)$ by $\hat{\Gamma}$. Hence $\mathrm{Pic}(T)$ can be identified with the group $(K^*)^n/\hat{\Gamma}$ which we denote by $\hat{T}$.

THEOREM 3.11. *There is a Zariski dense set X in $\hat{T}$ such that whenever $d \in X$ we can choose the corresponding cohomology class $c_d \in \mathrm{Hom}$ (Γ, K^*) which takes values in the group of units U.*

Proof. Consider the exact sequence,

$$0 \to U \to K^* \xrightarrow{ord} \mathbb{Z} \to 0$$

where ord is the order function and U is the group of units. This induces

$$0 \to \mathrm{Hom}(\Gamma, U) \to \mathrm{Hom}(\Gamma, K^*) \to \mathrm{Hom}(\Gamma, \mathbb{Z}) \to 0.$$

If we identify $\mathrm{Hom}(\Gamma, K^*)$ with $(K^*)^n$ then by the Lemma 3.10, we see that the image of $\mathrm{Hom}(\Gamma, U)(\cong U^n)$ is Zariski dense in $(K^*)^n$. Hence the image of $\mathrm{Hom}(\Gamma, U)$ in $(K^*)^n/\hat{\Gamma}$ is Zariski dense. □

References

[1] L. Gerritzen, *On nonarchimedean representations of abelian varieties*, Math. Ann. **196** (1972), 323-346.

[2] U. Köpf, *Über ergentliche Familien algebraischer Varietäten über affinoiden Räumen*, Schriftereihe Math. Inst. Univ. Münster, 2. Serie, Heft 7.

[3] S. Lang, *Fundamental of Diophantine geometry*, Springer Verlag, New York, 1961.

[4] A. Robert, *Elliptic curves*, Lecture Notes in Math, Berlin, Heidelberg, New York, Springer **326** (1973).

[5] H. Tapia-Recillas, *Ultrametric theta functions and abelian varieties*, Nagoya Math. J. **69** (1978), 65-96.

[6] S. Turner, *Nonarchimedean theta function and the Néron symbol*, J. Number Theory, **21** (1985), 1-16.

[7] A. Werner, *Local height on abelian varieties with split multiplicative reduction*, Compositio Mathematica, **107** (1997), 289-317.

KYUNG AH SHIM AND SUNG SIK WOO, DEPARTMENT OF MATHEMATICS, EWHA WOMANS UNIVERSITY, SEOUL, 120-750, KOREA

E-mail: Shimkah@hanmail.net and sswoo@mm.ewha.ac.kr