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Fuzzy Linear Regression Model Using the Least Hausdorff-distance Square Method

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Abstract

In this paper, we review some class of t-norms on which fuzzy arithmetic operations preserve the shapes of fuzzy numbers and the Hausdorff-distance between fuzzy numbers as the measure of distance between fuzzy numbers. And we suggest the least Hausdorff-distance square method for fuzzy linear regression model using shape preserving fuzzy arithmetic operations.

Keywords : Fuzzy number, The weakest t-norm, The least Hausdorff-distance square method.

1. Introduction

Linear regression models are widely used today in business, administration, economics, engineering, as well as in many other traditionally non-quantitative fields such as social, health, and biological sciences. In all cases of fuzzy regression, the linear regression is recommended for practical situations when decisions often have to be made on the basis of imprecise and/or partially available data. Many different fuzzy regression approaches have been proposed. Fuzzy regression, as first developed by Tanaka et al.(1980) in a linear system, is based on the extension principle. Tanaka et al.(1980) initially applied their fuzzy linear regression procedure to non-fuzzy experimental data. In the experiments that followed this pioneering effort Tanaka et al.(1982) used fuzzy input experimental data to build fuzzy regression models. Fuzzy input data used in these experiments were given in the form of triangular fuzzy numbers. The process was explained in more detail by Dubois and Prade(1980). A technique for linear least-square fitting of several fuzzy variable was developed by Diamond(1988) giving the solution to an analog of the normal equation of classical least squares. Bardossy(1990) expanded the methodology to include different vagueness criteria and

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provide answers for a non-linear relationship between model variables. Wang and Li(1990) studied maximum μ/E estimation of the parameters of the distribution of the possibility variables and presented two different fuzzy linear regression models of fuzzy variables using maximum μ/E estimation and entropy of possibility variables. In addition to the above, fuzzy maximum-likelihood regression has been presented by Okuda et al.(1992). This method develops the maximum-likelihood estimates of the model's parameters by the usual statistical method under fuzzy observations. Furthermore, Sakawa and Yano(1992) proposed a multiobjective programing approach for obtaining fuzzy linear regression models. The application of the statistical technique for fuzzy numbers was introduced by Viertl(1996). The comparison between fuzzy and statistic theory has been presented by Kim et al.(1999).

In this paper, we introduce some class of shape preserving fuzzy arithmetic operations based on sup-t-norm convolution. Hausdorff measure of distance between fuzzy numbers is defined and then a least-square optimization is performed to estimate the regression parameters. It is noted that Diamond(1988) used a different distance measure and proposed so-called fuzzy least squares. Some examples on estimations of parameters based on T_M =Min, T_W=the weakest t-norm and T_P=Yager's t-norm are given.

2. Preliminaries

Definition 2.1. Let R be the real number field. $\mu : R \rightarrow [0,\ 1]$ is called a fuzzy number if

(1) $\forall \alpha \in (0, 1]$, $\mu_\alpha = \{x;\ \mu(x)\ \geq\ \alpha\}$ is a finite closed interval.

(2) $\mu_1 = \{x : \mu(x) = 1\} \neq \emptyset$.

Note: $F(R)$ is the set of all fuzzy numbers.

Definition 2.2. For $\mu \in F(R)$, μ is called symmetric if

$$\exists x_0 \in R \ \cdot \ni \cdot \ \mu(x_0 + x) = \mu(x_0 - x),\ \forall x \in R.$$

In this cases, x_0 is called mean of μ. If $x_0 = 0$, then μ is called 0-symmetric and denoted by $\overline{\mu}$.

Definition 2.3. Let $\overline{\mu}$ be a 0-symmetric fuzzy number; $L_{\overline{\mu}}$ is the family of fuzzy numbers generated by $\overline{\mu}$ if

$$L_{\overline{\mu}} = \left\{ Q_{\overline{\mu}}(a, b) \mid Q_{\overline{\mu}} = \overline{\mu}\left(\frac{x-a}{b}\right),\ a \in R,\ b \in R_+ \right\},$$

where $R_+ = [0, +\infty)$. If $b = 0$, then $\overline{\mu}\left(\frac{x-a}{b}\right) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{otherwise}. \end{cases}$

In this, it is said that a is a center and b is a width.

Definition 2.4. Let $X=[0,1]$, V is the Cartesian product of X. A t-norm is a function T from V into X which satisfies the following:

(1) $T(x,0)=0,\ T(x,1)=x$,

(2) $T(x,y)=T(y,x)$,

(3) $(x\le x',\ y\le y')\to T(x,y)\le T(x',y')$,

(4) $T(T(x,y),z)=T(x,T(y,z))$.

Also, it can be extended to n dimension by deductive method and denoted by $\underset{i=1,\cdots,n}{T}(x_i)=T(x_1,\cdots,x_n)=T(T(T(T(x_1,x_2),x_3),\cdots,x_{n-1}),x_n)$.

Example 2.1.

(1) $T_M(\mu_1(x_1),\cdots,\mu_n(x_n))=\min\{\mu_1(x_1),\cdots,\mu_n(x_n)\}$ is called the minimum t-norm, denoted by T_M.

(2) $$T_W(\mu_1(x_1),\cdots,\mu_n(x_n))=\begin{cases}\mu_1(x_1) & \text{if } \mu_2(x_2)=,\cdots,=\mu_n(x_n)=1,\\ \vdots & \\ \mu_n(x_n) & \text{if } \mu_1(x_1)=,\cdots,=\mu_{n-1}(x_{n-1})=1,\\ 0 & \text{otherwise.}\end{cases}$$

is called the weakest t-norm, denoted by T_W.

(3) $$T_P(\mu_1(x_1),\cdots,\mu_n(x_n)) = \max\left\{0,\ 1-\left(\sum_{i=1}^{n}(1-\mu_i(x_i))^p\right)^{1/p}\right\},$$

where $1/p+1/q=1$, is called the parameterized Yager t-norm, denoted by T_P.

Definition 2.5. Let $f: R^n\to R$ be a real-valued mapping and $\mu_1,\cdots,\mu_n$ be n fuzzy numbers. Then the extension principle(sup-t-norm convolution) defines

$$f(\mu_1,\cdots,\mu_n)(y)=\sup\{T(\mu_1(x_1),\cdots,\mu_n(x_n));(x_1,\cdots,x_n)\in R^n,\ y=f(x_1,\cdots,x_n)\},$$

where T is an arbitrary t-norm.

Property 2.1. [5, 9] Let $Q_{\bar{\mu}}(a_i,b_i),\ Q_{\bar{\mu}}(c_i,d_i)\in L_{\bar{\mu}},\ e_i\in R$, and $i=1,\cdots,n$.

If t-norm is T_M, then

$$\sum_{i=1}^{n} e_i Q_{\bar{\mu}}(a_i,b_i) = Q_{\bar{\mu}}\left(\sum_{i=1}^{n} e_i a_i,\ \sum_{i=1}^{n}|e_i|b_i\right)\in L_{\bar{\mu}}.$$

If t-norm is T_W, then

$$\sum_{i=1}^{n} e_i Q_{\bar{\mu}}(a_i,b_i) = Q_{\bar{\mu}}\left(\sum_{i=1}^{n} e_i a_i,\ \max_{i=1,\cdots,n}\{|e_i|b_i\}\right)\in L_{\bar{\mu}}.$$

and

$$\sum_{i=1}^{n} Q_{\bar{\mu}}(a_i, b_i)Q_{\bar{\mu}}(c_i, d_i) = Q_{\bar{\mu}}\Big(\sum_{i=1}^{n} a_i c_i, \max_{i=1,\cdots,n} \{|a_i|d_i, |c_i|b_i\}\Big) \in L_{\bar{\mu}}.$$

If t-norm is T_P, then

$$\sum_{i=1}^{n} e_i Q_{\bar{\mu}}(a_i, b_i) = Q_{\bar{\mu}}\Big(\sum_{i=1}^{n} e_i a_i, \big(\sum_{i=1}^{n}(|e_i|b_i)^q\big)^{1/q}\Big) \in L_{\bar{\mu}}.$$

with the extreme cases;

- $q=1(p \to \infty)$ (min-operator);

$$\sum_{i=1}^{n} e_i Q_{\bar{\mu}}(a_i, b_i) = Q_{\bar{\mu}}\Big(\sum_{i=1}^{n} e_i a_i, \sum_{i=1}^{n} |e_i| b_i\Big) \in L_{\bar{\mu}}.$$

- $p=1(q \to \infty)$ (bounded difference);

$$\sum_{i=1}^{n} e_i Q_{\bar{\mu}}(a_i, b_i) = Q_{\bar{\mu}}\Big(\sum_{i=1}^{n} e_i a_i, \max_{i=1,\cdots,n} \{|e_i|b_i\}\Big) \in L_{\bar{\mu}}.$$

Example 2.2. Consider

$$Tri(x) = \begin{cases} 1+x & \text{if } x \in [-1,0], \\ 1-x & \text{if } x \in (0,1], \\ 0 & \text{otherwise}, \end{cases}$$

$$Nor(x) = e^{-\pi x^2},$$

and

$$Par(x) = \begin{cases} 1-x^2 & \text{if } x \in [-1,1], \\ 0 & \text{otherwise}. \end{cases}$$

These are 0-symmetric fuzzy number and the family of fuzzy numbers generated by *Tri*, *Nor*, and *Par* are called trigonometric, normal, and parabolic respectively. Write $Tri(a,b) = Q_{Tri}(a,b)$, $Nor(a,b) = Q_{Nor}(a,b)$, and $Par(a,b) = Q_{Par}(a,b)$.

Definition 2.6. Let A, B the compact subsets in R. Let

$$h(A,B) = \max\Big\{ \sup_{a\in A} \inf_{b\in B} \| a-b \|, \sup_{b\in B} \inf_{a\in A} \| a-b \| \Big\},$$

where A, $B \neq \emptyset$. Then h is the Hausdorff-metric.

Definition 2.7. Let $\mu_1, \mu_2 \in F(R)$. We call that

$$H(\mu_1, \mu_2) = \int_0^1 h(\mu_{1\alpha}, \mu_{2\alpha})\, d\alpha$$

is the Hausdorff-distance between fuzzy numbers.

Property 2.2. [7] Let $\mu_i = Q_{\bar{\mu}}(a_i, b_i)$, $i = 1,2$, where $\bar{\mu}$ is a 0-symmetric fuzzy number. Then

$$H(\mu_1, \mu_2) = |a_1 - a_2| + l|b_1 - b_2|, \quad \text{where } l = \int_0^1 | \overline{u}^{-1}(\alpha)| d\alpha$$

In this case, l is exactly the half of $\int_R \overline{\mu}(x)\, dx$.

Example 2.3. If $\overline{\mu}$ is trigonometric, then $\int_R \overline{\mu}(x)\, dx = 1$.

If $\overline{\mu}$ is normal, then $\int_R \overline{\mu}(x)\, dx = 1$.

If $\overline{\mu}$ is parabolic, then $\int_R \overline{\mu}(x)\, dx = \frac{4}{3}$.

3. Fuzzy linear regression model using the least Hausdorff-distance square method

This section considers the linear regression model of fuzzy numbers, that is, a model with k regressor variables $X_1, \cdots, X_k$ that has a relationship with a response variable Y that is a straight line. This linear regression model is

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k,$$

where the coefficients $\beta_0, \beta_1, \cdots, \beta_k$ are unknown constants(or fuzzy numbers) and $Y, X_1, \cdots, X_k$ are fuzzy numbers. Now our problem is how to optimally determine the coefficients $\beta_i (i = 0, 1, \cdots, k)$ by observing fuzzy numbers. It is considered in this section that fuzzy numbers are limited to some class of fuzzy numbers $L_{\overline{\mu}}$.

3.1 Estimation of coefficients in fuzzy linear regression model using the least Hausdorff-distance square method.

Let $X_{ij} = Q_{\overline{\mu}}(a_{ij}, b_{ij})$ and $Y_j = Q_{\overline{\mu}}(c_j, d_j)$. Write $\overline{Y_j} = \beta_0 + \beta_1 X_{1j} + \cdots + \beta_k X_{kj}$, $j = 1, \cdots, n$. Like referring to earlier, we can have t-norm such that $\overline{Y_j} = Q_{\overline{\mu}}(e_j, f_j)$, in that e_j and f_j are the function of coefficients. For instance, there are the minimum t-norm, the weakest t-norm, and the parameterized Yager t-norm. Then $\overline{Y_j}$ is the estimated fuzzy number of Y_j. Thus, if one wishes to fit a straight line through a set of fuzzy numbers, the equation $Y = \widehat{\beta}_0 + \widehat{\beta}_1 X_1 + \cdots + \widehat{\beta}_k X_k$ provides a straight line that minimizes the sum of squares of the Hausdorff-distance errors between the observed fuzzy numbers and the estimated fuzzy numbers, say

$$SSHE(\widehat{\beta}_0, \cdots, \widehat{\beta}_k) = \sum_{j=1}^{n} h^2(Y_j - \overline{Y_j}).$$

The quantities *SSHE* is called the Hausdorff-distance error sum of square. This method is called the least Hausdorff-distance square method. and $\widehat{\beta_1}, \cdots, \widehat{\beta_k}$ is called the least Hausdorff-distance square estimation(MHE) of $\beta_1, \cdots, \beta_k$.

3.2 Examples

We now consider some illustrative examples for above models.

Example 3.1. We take initial data from the Table 1.

Table 1. Initial data.

Y	X_1
Y_1 = (2.5, 0.38)	X_{11} = (6.0, 0.6)
Y_2 = (2.0, 0.4)	X_{12} = (9.0, 1.35)
Y_3 = (3.0, 0.45)	X_{13} = (12.0, 1.2)
Y_4 = (2.5, 0.5)	X_{14} = (12.0, 2.4)
Y_5 = (3.5, 0.35)	X_{15} = (15.0, 1.5)
Y_6 = (3.0, 0.3)	X_{16} = (15.0, 2.25)
Y_7 = (3.5, 0.7)	X_{17} = (18.0, 3.6)
Y_8 = (4.0, 0.8)	X_{18} = (21.0, 2.1)

If $T = T_M$, $X_{ij} = Q_{\bar{\mu}}(a_{ij}, b_{ij})$, $Y_j = Q_{\bar{\mu}}(c_j, d_j)$, and $\beta_0, \cdots, \beta_k \in R$, then by Property 2.1

$$\overline{Y_j} = Q_{\bar{\mu}}\left(\sum_{i=0}^{k} a_{ij}\beta_i, \ \sum_{i=0}^{k} b_{ij}|\beta_i| \right),$$

where $a_{0j} = 1$ and $b_{0j} = 0$.

We estimate β_0 and β_1 by least Hausdorff-distance square method. Now,

$$SSHE(\beta_0, \beta_1) = \sum_{j=1}^{8} (|c_j - (\beta_0 + \beta_1 \times a_{1j})| + l|d_j - |\beta_1| \times b_{1j}|)^2,$$

where $l = \int_0^1 |\bar{u}^{-1}(\alpha)| d\alpha$. We have to choose $\widehat{\beta_0}$ and $\widehat{\beta_1}$ such that $SSHE(\widehat{\beta_0}, \widehat{\beta_1}) = \min_{\beta_0, \beta_1} SSHE(\beta_0, \beta_1)$. In this cases, if $\bar{\mu} = Tri$, then $l = 1/2$ and the estimated parameters is $\widehat{\beta_0} = 1.249$, $\widehat{\beta_1} = 0.131$, and $SSHE(\widehat{\beta_0}, \widehat{\beta_1}) = 1.244$.

If $T = T_M$, $X_{ij} = Q_{\bar{\mu}}(a_{ij}, b_{ij})$, $Y_j = Q_{\bar{\mu}}(c_j, d_j)$, $\beta_0 = Q_{\bar{\mu}}(e_0, f_0)$, and $\beta_1, \cdots, \beta_k \in R$, then by Property 2.1

$$\overline{Y_j} = Q_{\overline{\mu}}\left(e_0 + \sum_{i=1}^{k} a_{ij}\beta_i,\ f_0 + \sum_{i=1}^{k} b_{ij}|\beta_i|\right).$$

We estimate β_0 and β_1 by least Hausdorff-distance square method. Now,

$$SSHE(\beta_0, \beta_1) = \sum_{j=1}^{8}(|c_j - (e_0 + \beta_1 \times a_{1j})| + l|d_j - (f_0 + |\beta_1| \times b_{1j})|)^2,$$

where $l = \int_0^1 |\ \overline{u}^{-1}(\alpha)|d\alpha$. We have to choose $\widehat{e_0}$, $\widehat{f_0}$, and $\widehat{\beta_1}$ such that $SSHE(\widehat{e_0}, \widehat{f_0}, \widehat{\beta_1}) = \min_{e_0, f_0, \beta_1} SSHE(e_0, f_0, \beta_1)$. In this cases, if $\overline{\mu} = Tri$, then $l = 1/2$ and the estimated parameters is $\widehat{\beta_0} = (1.355,\ 0.234)$, $\widehat{\beta_1} = 0.123$, and $SSHE(\widehat{\beta_0}, \widehat{\beta_1}) = 0.829$.

Example 3.2. We take the initial data from the Table 1 in Example 3.1. If $T = T_{P=5}$, $X_{ij} = Q_{\overline{\mu}}(a_{ij}, b_{ij})$, $Y_j = Q_{\overline{\mu}}(c_j, d_j)$, and $\beta_0, \cdots, \beta_k \in R$, then by Property 2.1

$$\overline{Y_j} = Q_{\overline{\mu}}\left(\sum_{i=0}^{k} a_{ij}\beta_i,\ \left(\sum_{i=0}^{k}(b_{ij}|\beta_i|)^5\right)^{\frac{1}{5}}\right),$$

where $a_{0j} = 1$ and $b_{0j} = 0$.

We estimate β_0 and β_1 by least Hausdorff-distance square method. Now,

$$SSHE(\beta_0, \beta_1) = \sum_{j=1}^{8}(|c_j - (\beta_0 + \beta_1 \times a_{1j})| + l|d_j - |\beta_1| \times b_{1j}|)^2,$$

where $l = \int_0^1 |\ \overline{u}^{-1}(\alpha)|d\alpha$. We have to choose $\widehat{\beta_0}$ and $\widehat{\beta_1}$ such that $SSHE(\widehat{\beta_0}, \widehat{\beta_1}) = \min_{\beta_0, \beta_1} SSHE(\beta_0, \beta_1)$. In this cases, if $\overline{\mu} = Tri$, then $l = 1/2$ and the estimated parameters is $\widehat{\beta_0} = 1.249$, $\widehat{\beta_1} = 0.131$, and $SSHE(\widehat{\beta_0}, \widehat{\beta_1}) = 1.244$.

If $T = T_{P=5}$, $X_{ij} = Q_{\overline{\mu}}(a_{ij}, b_{ij})$, $Y_j = Q_{\overline{\mu}}(c_j, d_j)$, $\beta_0 = Q_{\overline{\mu}}(e_0, f_0)$, and $\beta_1, \cdots, \beta_k \in R$, then by Property 2.1

$$\overline{Y_j} = Q_{\overline{\mu}}\left(e_0 + \sum_{i=1}^{k} a_{ij}\beta_i,\ \left(f_0^5 + \sum_{i=1}^{k}(b_{ij}|\beta_i|)^5\right)^{\frac{1}{5}}\right).$$

We estimate β_0 and β_1 by least Hausdorff-distance square method. Now,

$$SSHE(\beta_0, \beta_1) = \sum_{j=1}^{8}(|c_j - (e_0 + \beta_1 \times a_{1j})| + l|d_j - (f_0^5 + |\beta_1|^5 \times b_{1j}^5)^{\frac{1}{5}}|)^2,$$

where $l = \int_0^1 |\ \overline{u}^{-1}(\alpha)|d\alpha$. We have to choose $\widehat{e_0}$, $\widehat{f_0}$, and $\widehat{\beta_1}$ such that $SSHE(\widehat{e_0}, \widehat{f_0}, \widehat{\beta_1}) = \min_{e_0, f_0, \beta_1} SSHE(e_0, f_0, \beta_1)$. In this cases, if $\overline{\mu} = Tri$, then $l = 1/2$ and the estimated parameters is $\widehat{\beta}_0=(1.279,\ 0.399)$, $\widehat{\beta}_1=0.128$, and $SSHE(\widehat{\beta}_0, \widehat{\beta}_1)=0.828$.

Example 3.3. We take the initial data from the Table 1 in Example 3.1. If $T=T_W$, $X_{ij}=Q_{\overline{\mu}}(a_{ij}, b_{ij})$, $Y_j=Q_{\overline{\mu}}(c_j, d_j)$, and $\beta_0,\cdots,\beta_k\in R$, then by Property 2.1

$$\overline{Y_j}=Q_{\overline{\mu}}\left(\sum_{i=0}^{k} a_{ij}\beta_i,\ \max_{i=0,\cdots,k}\{b_{ij}|\beta_i|\}\right),$$

where $a_{0j}=1$ and $b_{0j}=0$.

We estimate β_0 and β_1 by least Hausdorff-distance square method. Now,

$$SSHE(\beta_0,\beta_1)=\sum_{j=1}^{8}(|c_j-(\beta_0+\beta_1\times a_{1j})|+l|d_j-|\beta_1|\times b_{1j}|)^2,$$

where $l=\int_0^1|\overline{u}^{-1}(\alpha)|d\alpha$. We have to choose $\widehat{\beta}_0$ and $\widehat{\beta}_1$ such that $SSHE(\widehat{\beta}_0, \widehat{\beta}_1)=\min_{\beta_0,\beta_1} SSHE(\beta_0,\beta_1)$. In this cases, if $\overline{\mu}=Tri$, then $l=1/2$ and the estimated parameters is $\widehat{\beta}_0=1.249$, $\widehat{\beta}_1=0.131$, and $SSHE(\widehat{\beta}_0, \widehat{\beta}_1)=1.244$.

If $T=T_W$, $X_{ij}=Q_{\overline{\mu}}(a_{ij}, b_{ij})$, $Y_j=Q_{\overline{\mu}}(c_j, d_j)$, $\beta_0=Q_{\overline{\mu}}(e_0, f_0)$, and $\beta_1,\cdots,\beta_k\in R$, then by Property 2.1

$$\overline{Y_j}=Q_{\overline{\mu}}\left(e_0+\sum_{i=1}^{k} a_{ij}\beta_i,\ \max\left\{f_0,\ \max_{i=1,\cdots,k}\{b_{ij}|\beta_i|\}\right\}\right).$$

We estimate β_0 and β_1 by least Hausdorff-distance square method. Now,

$$SSHE(\beta_0,\beta_1)=\sum_{j=1}^{8}(|c_j-(e_0+\beta_1\times a_{1j})|+l|d_j-\max\{f_0,|\beta_1|\times b_{1j}\}|)^2,$$

where $l=\int_0^1|\overline{u}^{-1}(\alpha)|d\alpha$. We have to choose $\widehat{e}_0$, $\widehat{f}_0$, and $\widehat{\beta}_1$ such that $SSHE(\widehat{e}_0, \widehat{f}_0, \widehat{\beta}_1)=\min_{e_0,f_0,\beta_1} SSHE(e_0,f_0,\beta_1)$. In this cases, if $\overline{\mu}=Tri$, then $l=1/2$ and the estimated parameters is $\widehat{\beta}_0=(1.277,\ 0.400)$, $\widehat{\beta}_1=0.128$, and $SSHE(\widehat{\beta}_0, \widehat{\beta}_1)=0.841$.

If $T=T_W$, $X_{ij}=Q_{\overline{\mu}}(a_{ij}, b_{ij})$, $Y_j=Q_{\overline{\mu}}(c_j, d_j)$, $\beta_0=Q_{\overline{\mu}}(e_0, f_0)$, and $\beta_i=Q_{\overline{\mu}}(e_i, f_i)$, then by Property 2.1

$$\overline{Y_j}=Q_{\overline{\mu}}\left(\sum_{i=0}^{k} a_{ij}e_i,\ \max_{i=0,\cdots,k}\{|a_{ij}|f_i, |e_i|b_{ij}\}\right),$$

where $a_{0j}=1$ and $b_{0j}=0$.

We estimate β_0 and β_1 by least Hausdorff-distance square method. Now,

$$SSHE(\beta_0, \beta_1) = \sum_{j=1}^{8}(|c_j - (e_0 + e_1 \times a_{1j})| + l|d_j - \max\{f_0, |a_{1j}|f_1, |e_1|b_{1j}\}|)^2,$$

where $l = \int_0^1 |\overline{u}^{-1}(\alpha)|d\alpha$. We have to choose $\widehat{e_0}$, $\widehat{f_0}$, $\widehat{e_1}$, and $\widehat{f_1}$ such that $SSHE(\widehat{e_0}, \widehat{f_0}, \widehat{e_1}, \widehat{f_1}) = \min_{e_0, f_0, e_1, f_1} SSHE(e_0, f_0, e_1, f_1)$. In this cases, if $\overline{\mu} = Tri$, then $l = 1/2$ and the estimated parameters is $\widehat{\beta_0} = (1.363, 0.400)$, and $\widehat{\beta_1} = (0.121, 0.0267)$.

Figure 1. The fuzzy linear regression model using T_W with real β_0 and real β_1.

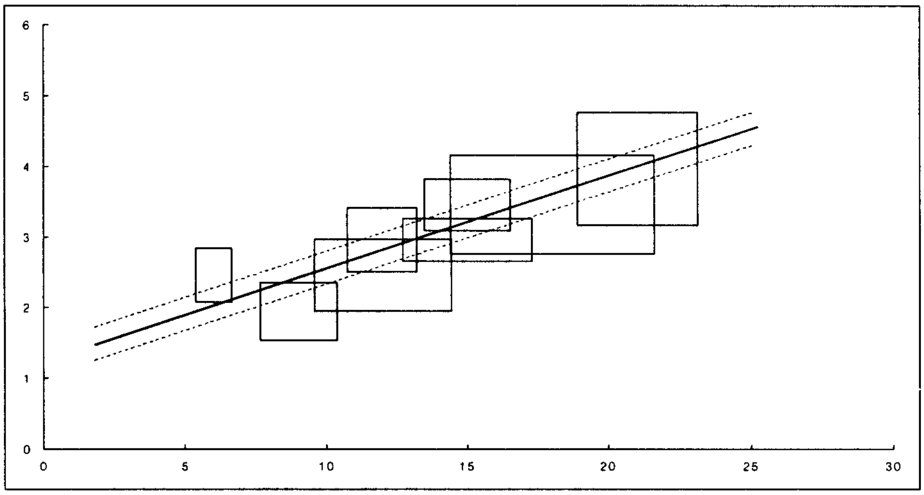


Figure 2. The fuzzy linear regression model using T_W with fuzzy β_0 and real β_1.

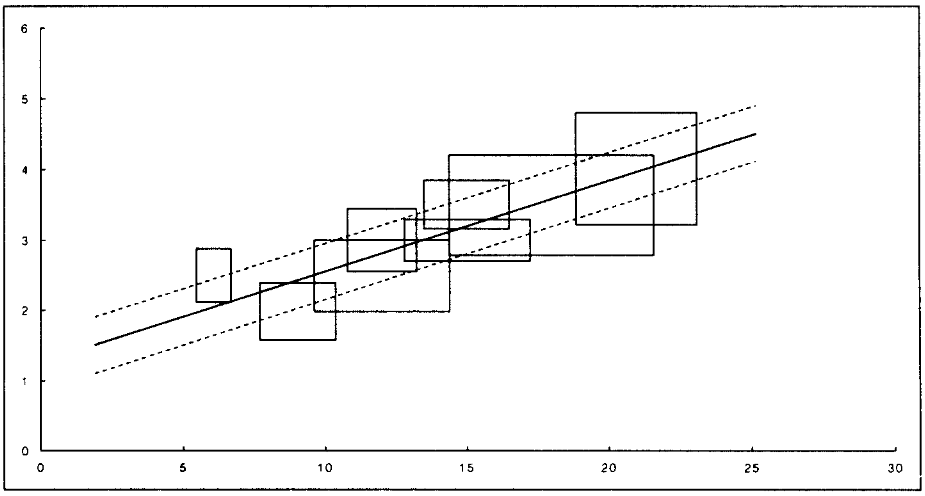
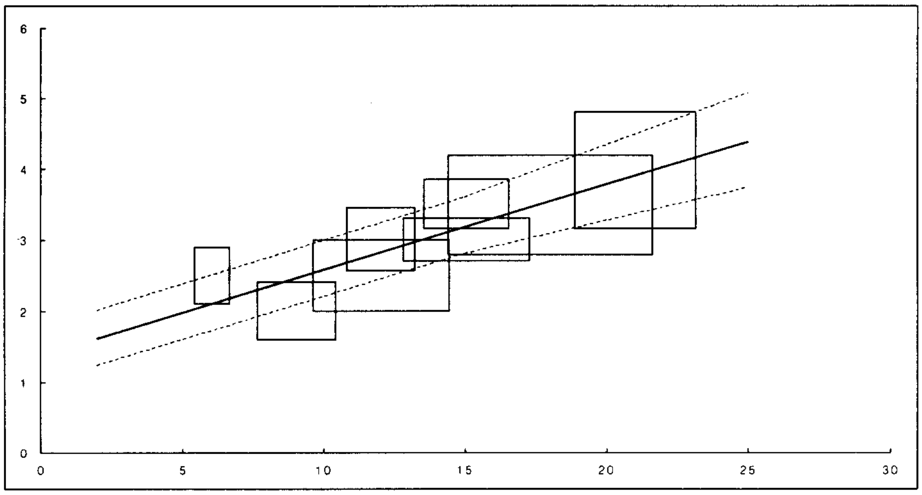


Figure 3. The fuzzy linear regression model using T_W with fuzzy $\beta_0$and fuzzy β_1.



In Figure 1, Figure 2, and Figure 3, we draw them using the result of Example 3.3. In Figure 1, β_0 and β_1 are all constants. In Figure 2, β_0 is a fuzzy number and β_1 is a constant. In Figure 3, β_0 and β_1 are all fuzzy numbers. the eight rectangles are loci of the membership functions of the fuzzy input-output data (X_{1i}, Y_i), $i=1,\cdots,8$, whose values become zero. The straight line and two brcken lines are loci of the membership function of the fuzzy linear regression model where the width of X_1 is $\sum_{i=1}^{8} b_{1i}/8 = 1.875$.

Table 2. Compared table

t-norm	$\widehat{\beta_0}$	$\widehat{\beta_1}$	$SSHE$
T_M	1.249	0.131	1.244
	(1.355, 0.234)	0.123	0.829
$T_{P=5}$	1.249	0.131	1.244
	(1.279, 0.399)	0.128	0.828
T_W	1.249	0.131	1.244
	(1.277, 0.400)	0.128	0.841
	(1.363, 0.400)	(0.121, 0.0267)	0.801

The summarized result of Example 3.1, 3.2, and 3.3 are shown in Table 2. Comparing the estimation, the value $SSHE$ is the smallest in case $t=T_W$ and β_0 and β_1 are all fuzzy numbers. But, for the cases where real β_0, real β_1, and fuzzy β_0, real β_1, the results are very similar even though t-norm are different.

If we separately inspect the center and width of observed fuzzy numbers in example then the slope of center using the linear regression in statistics is 0.12037 and the mean width of the response is 0.2587 times of that of the regressor. Above-mentioned, the linearity of center and width exists but the slopes of the linear model are different. Looking over the Figure 1, the estimated width of Y doesn't sufficiently express the observed width of Y. Because the estimated slope of $\widehat{\beta}_1 = 0.131$ is much less than the slope of width of 0.2587. $\widehat{\beta}_1 = 0.131$ is slightly more than the slope of center of 0.12037 as we expect. This is a indecisive result as we excessively use the one constant to explain the linearity of center and width. In Figure 2, the estimated fuzzy linear regression model express the observed fuzzy data on the whole. But $\widehat{\beta}_1 = 0.128$ is slightly interfered by the slope of width of 0.2587. On the other hand, in Figure 3, the estimated fuzzy linear regression model well expresses the observed fuzzy data and the interference isn't exist. Hence it is better using the fuzzy number to estimate the coefficient.

4 Conclusion

In this paper, we have presented a new method to evaluate fuzzy linear regression models using shape preserving t-norm-based fuzzy arithmetic operations where both input data and output data are fuzzy numbers. For the cases where real β_0, real β_1, or fuzzy β_0, real β_1, the value $SSHE$ and the coefficients are very similar even though t-norms are different. As we saw in this paper, T_W-based fuzzy arithmetic operations (addition and multiplication) preserves the shape of fuzzy numbers. Because of this advantage, we can take β_0 and β_1 to be fuzzy numbers, and the resulting $SSHE$ is the smallest.

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