The Korean Communications
in Statistics Vol. 7, No. 3, 2000
pp. 955-958

A Note on Spacings

S. H. Kim[1)]

Abstract

In this paper, it will be shown that if the distribution function F of X is increasing (decreasing) failure rate, then the spacings are negatively (positively) dependent. Some numerical examples are illustrated.

Keywords : Spacing, Order statistics, IFR, DFIR

1. Introduction

Let $X_1, \ldots, X_n$ be independent and identically distributed variates and let $X_{1:n} \leq \ldots \leq X_{n:n}$ denote the order statistics. The spacings are defined as $D_r = X_{r:n} - X_{r-1:n}$. If X is distributed exponentially, then spacings $D_1, D_2, \ldots, D_n$ are mutually independent and each D_r is distributed exponentially (see, e.g., David(1981, p. 21)).

Spacings has been reviewed for both small (David(1981, Section 5.4)) and large samples by Pyke(1965) and Wells(1993), who give many references.

For our convenience, we state here some well-known definitions (see, e.g., Barlow and Proschan, 1975).

Definition 1. The distribution F of X is said to have increasing (decreasing) failure rate if for $t>0$,

$$\frac{1-F(x+t)}{1-F(x)} \text{ is decreasing (increasing) in } x \text{ for each } t \geq 0. \qquad (1)$$

More simply we say that F is IFR (DFR). When the pdf $f(x)$ exists, (1) is equivalent to saying that the hazard rate $r(x)=f(x)/[1-F(x)]$ increases with x (Barlow and Proschan,

1) Associate Professor, Department of Statistics and Information, Anyang University, Anyagn-shi 430-714, Korea.
E-mail: shkim@aycc.anyang.ac.kr

1975, p. 54).

Definition 2. Given rv's X and Y, we say that Y is stochastically increasing (decreasing) in X if $P(Y > y | X = x)$ is increasing (decreasing) in x for all y.
It is noted that if Y is stochastically increasing (decreasing) in X, then $cov(X, Y) \geq (\leq) 0$. Throughout this paper, increasing is short for nondecreasing and decreasing is short for nonincresing.

In section 2, we will show that if the distribution function F of X is IFR (DFR), then the spacings are negatively (positively) dependent. Some numerical illustrations are given by section 3.

2. Dependence on Spacings

The proof of following Lemma used the preliminary result of Tukey(1958) and the Markov property of order statistics (e.g., David, 1981, p. 20).

Lemma [Kim and David(1990)]. If the distribution F of X is IFR (DFR), then $X_{s:n} - X_{r:n}$ is stochastically decreasing (increasing) in $X_{k:n}$ for any k satisfying $i \leq k \leq r < s \leq n$.

Now using Lemma, we will show the dependence structure of spacings with respect to monotone failure rate classes.

Theorem. If the distribution F of X is IFR, then D_s is stochastically decreasing on D_r (this implies $cov(D_r, D_s) \leq 0$ for any $1 \leq r < s \leq n$).

Proof.

$$P\{D_s > w | D_r = x\} = P\{X_{s:n} - X_{s-1:n} > w | X_{r:n} - X_{r-1:n} = x\}$$

$$= \int_{-\infty}^{\infty} P\{X_{s:n} - X_{s-1:n} > w | X_{r:n} = x + y, X_{r-1:n} = y\} dF_{r-1:n}(y). \quad (2)$$

From the Markov property of order statistics (see, e.g., David (1981, p. 20)),

$$f_{X_{s:n} | X_{r:n} = x_{(r)},\ X_{k:n} = x_{(k)}}(y) = f_{X_{s:n} | X_{r:n} = x_{(r)}}(y). \quad (3)$$

Using (3), (2) can be expressed by,

$$= \int_{-\infty}^{\infty} P\{X_{s:n} - X_{s-1:n} > w | X_{r:n} = x + y\} dF_{r-1:n}(y). \qquad (4)$$

If F is IFR, then by Lemma, for any $0 < x_1 < x_2$, and any fixed y,

$$P\{X_{s:n} - X_{s-1:n} > w | X_{r:n} = x_2 + y\} - P\{X_{s:n} - X_{s-1:n} > w | X_{r:n} = x_1 + y\} \le 0 . \qquad (5)$$

From (5), $P\{D_s > w | D_r = x\}$ is decreasing in x for any w, i.e., D_s is stochastically decreasing on D_r . ■

Similarly, it follows that if F is DFR, D_s is stochastically increasing on D_r, which implies $cov(D_r, D_s) \ge 0$.

In life testing, a statistic that plays a central role is the total time on test. Assume n items are placed on test at time 0 and that successive failure are observed at times $X_{1:n} \le X_{2:n} \le . . \le X_{n:n}$. If we stop at the rth failure, then the total time on test is $\tau(t) = \sum_{i=1}^{r} (n-i+1) D_i$. If F is distributed exponentially, then

$$var\tau((t)) = \sum_{i=1}^{r} (n-i+1)^2 var(D_i).$$

Theorem shows that

$$var_{IFR}\, \tau((t)) \le var_{\text{EXP}}\, \tau((t)) \le var_{DFR}\, \tau((t)). \qquad (6)$$

3. Examples

Numerical illustrations are provided by the Gamma distribution, $f_r(x) = \frac{1}{\Gamma(r)} e^{-x} x^{r-1}$, for which, for $r > 1$, X is IFR (Table 1), and by the Pareto distribution for which X is DFR (Table 2). You can see Barlow and Proschan(1975) on the various classes of monotone failure rate density.

Table 1. Covariance of D_r and D_s in a sample of size 7 from Gamma(2,1) (Prescott(1974))

n	r	s	cov(D_r, D_s)	r	s	cov(D_r, D_s)
7	1	2	-0.0240	6	7	-0.0234
	2	3	-0.0100	1	4	-0.0078
	3	4	-0.0080	2	6	-0.0043
	4	5	-0.0083	3	7	-0.0053
	5	6	-0.0115	4	6	-0.0073

Table 2. Covariance of D_r and D_s in a sample of size 7 from $f(x)=3a^3x^{-4}$, a>0, x>=0 and and f(x)=0 otherwise (Malik(1966))

n	r	s	cov(D_r, D_s)	r	s	cov(D_r, D_s)
7	1	2	0.0001	6	7	0.0756
	2	3	0.0003	1	4	0.0003
	3	4	0.0008	2	6	0.0012
	4	5	0.0020	3	7	0.0058
	5	6	0.0079	4	6	0.0035

References

[1] Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston, New York.

[2] David, H.A. (1981). Order Statistics, 2nd edn. John Wiley and Sons, New York.

[3] Kim, S.H. and David, H.A. (1990). On the dependence structure of order statistics and concomitants of order statistics. Journal of Statistical Planning and Inference 24, 363-368.

[4] Malik, H.J. (1966) Exact moments of order statistics from the Pareto distribution. Skand. Aktuarietidskr. 49, 144-157.

[5] Prescott, P. (1974). Variances and covariances of order statistics from the gamma distribution. Biometrika 61, 607-613.

[6] Pyke, R. (1965) Spacings. J. R. Statist. Soc. B 27, 395-436. Discussion: 437-449.

[7] Tukey, J.W. (1958). A problem of Berkson, and minimum variance orderly estimators. Ann. Math. Statist. 29, 588-592.

[8] Wells, M.T. (1993) Large sample theory of spacings statistics for tests of fit for composite hypothesis, J. R. Statist. Soc. B 55(1), 189-203.