Bull. Korean Math. Soc. **38** (2001), No. 1, pp. 7–16

BOCHNER–SCHWARTZ THEOREM ON LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. We study the Fourier transformation on the Gelfand–Bruhat space of type S and characterize this space by means of Fourier transform on a locally compact abelian group G. Also, we extend Bochner–Schwartz theorem to the dual space of the Gelfand–Bruhat space and the space of Fourier hyperfunctions on G, respectively.

1. Introduction

Let G be an arbitrary locally compact abelian group. The theory of distributions on a group G has been introduced and studied extensively by F. Bruhat [1, 8], that is, he has extended the notion of a C^∞ function to a large class of a locally compact abelian group G. In particular, the Bochner–Schwartz theorem for distributions on a group G has been proved by K. Maurin [7] as follows.

THEOREM 1.1. *For a distribution $u \in \mathcal{D}'(G)$ on a group G the following are equivalent:*

(i) *u is positive definite, i.e., $\langle u, \varphi * \varphi^* \rangle \geq 0$ for every $\varphi \in \mathcal{D}(G)$, where $\varphi * \varphi^*(g) = \int_G \varphi(g+h)\overline{\varphi(h)}\, dh$ for any $g \in G$.*

(ii) *There exists a positive tempered* Borel *measure μ such that*

$$\langle u, \varphi \rangle = \int_\Gamma \widehat{\varphi}(\gamma)\, d\mu(\gamma), \quad \varphi \in \mathcal{D}(G),$$

where Γ is the group of characters of G.

This was extended by A. Wawrzyńczyk [10] to the space $\mathcal{S}'(G)$ of tempered distributions, who gave another equivalent definition of the Schwartz

Received November 19, 1999. Revised May 31, 2000.
2000 Mathematics Subject Classification: Primary 46F05; Secondary 43A25.
Key words and phrases: Bochner–Schwartz theorem, locally compact abelian group, positive definite, $\{M\}$-tempered.

space $\mathcal{S}(G)$ of rapidly decreasing functions on an arbitrary locally compact abelian group and proved the Bochner–Schwartz theorem for tempered distributions.

The aim of this paper is to define the Fourier transform of functions in the Gelfand–Bruhat space of type S on G and to extend the Bochner–Schwartz theorems to the dual space of the Gelfand–Bruhat space and the space of Fourier hyperfunctions on G, respectively. This representation is intrinsic to the group and makes no reference to Lie subquotients. Our proof of the main theorem makes use of topological properties of the Gelfand–Bruhat space on a group G and positive definiteness on elementary groups, which is quite different from the Wawrzyńczyk's proof and others.

2. Preliminaries

We first introduce an arbitrary locally compact abelian group. We refer to [10] for more details.

Elementary groups $\mathbb{R}^k \oplus \mathbb{T}^m \oplus \mathbb{Z}^\ell \oplus \mathbb{F}$ are building blocks of an arbitrary locally compact abelian group G, where $\mathbb{R}$ is the group of real numbers, $\mathbb{Z}$ is the group of integers, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the one dimensional torus, and $\mathbb{F}$ is a finite abelian group. A group G is said to be *compactly generated* if there is a compact neighborhood of zero such that G is generated by its elements.

Any group G is a direct limit of compactly generated groups G_λ associated with a continuous monomorphism h_λ, which are identical to the class of inverse limits of elementary groups G_λ^ν associated with a continuous epimorphism h_λ^ν, denoted by

$$G = \varinjlim_{\lambda} (G_\lambda, h_\lambda) = \varinjlim_{\lambda} \varprojlim_{\nu} (G_\lambda^\nu, h_\lambda^\nu) \big(= \varprojlim_{\nu} \varinjlim_{\lambda} (G_\lambda^\nu, h_\lambda^\nu)\big).$$

Since an arbitrary locally compact abelian group G is determined by elementary groups G_λ^ν, it is natural to study, first, the Fourier analysis on elementary groups.

A function $f(x, y, r, s)$, $(x, y) \in \mathbb{R}^k \oplus \mathbb{T}^m$, $(r, s) \in \mathbb{Z}^\ell \oplus \mathbb{F}$, on the elementary group will be called *differentiable* if for each (r, s) the function $f(\,\cdot\,, r, s)$ is differentiable on $\mathbb{R}^k \oplus \mathbb{T}^m$.

Now to introduce the Gelfand–Bruhat space of type S let M_p, $p = 0, 1, 2, \ldots$, be a sequence of positive numbers. Hereafter we always assume that a sequence M_p satisfies the following conditions throughout this paper.

(M.0) For some C, $L > 0$ $p! \leq CL^p M_p$, $p = 1, 2, \ldots$.

(M.1) $M_p^2 \leq M_{p-1}M_{p+1}$, $p = 1, 2, \ldots$.
(M.2) For some C, $H > 0$ $M_{p+q} \leq CH^{p+q}M_pM_q$, $p, q = 0, 1, 2, \ldots$.

REMARK. The sequence $M_p = (p!)^s$, $s \geq 1$, satisfies the above conditions, so does $M_p = m_2 \ldots m_p$ with $m_p = p(\log p)^r$, $r > 0$.

We also note that we do not assume the non-quasi-analyticity condition (M.3). Thus our results include both quasianalytic case and non quasianalytic case, in other words, both hyperfunctions and most of all ultradistributions.

For each sequence M_p of positive numbers we define its *associated function* $M(t)$ on $(0, \infty)$ by

$$M(t) = \sup_p \log \frac{t^p M_0}{M_p}.$$

For a constant $h > 0$ we define the Gelfand–Bruhat space of type S on locally compact abelian groups $\mathbb{T}$ and $\mathbb{Z}$. First, for $\mathbb{T}$ we write as follows.

$$\mathcal{S}_{M_p,h}(\mathbb{T}) = \Big\{\varphi \in C^\infty(\mathbb{T}) \;\Big|\; p_h(\varphi) = \sup_{\substack{t\in[0,2\pi]\\ k\in\mathbb{N}_0}} \frac{|\varphi^{(k)}(t)|}{h^k M_k} < \infty\Big\},$$

$\mathcal{S}_{\{M_p\}}(\mathbb{T}) = \varinjlim_{h\to\infty} \mathcal{S}_{M_p,h}(\mathbb{T})$, and $\mathcal{S}_{(M_p)}(\mathbb{T}) = \varprojlim_{h\to 0} \mathcal{S}_{M_p,h}(\mathbb{T})$. Also, for $\mathbb{Z}$ we write

$$\mathcal{S}_{M_p,h}(\mathbb{Z}) = \big\{\varphi : \mathbb{Z} \to \mathbb{C} \;\big|\; q_h(\varphi) = \sup_{n\in\mathbb{Z}} |\varphi(n)| e^{M(h|n|)} < \infty\big\},$$

$\mathcal{S}_{\{M_p\}}(\mathbb{Z}) = \varinjlim_{h\to 0} \mathcal{S}_{M_p,h}(\mathbb{Z})$, and $\mathcal{S}_{(M_p)}(\mathbb{Z}) = \varprojlim_{h\to\infty} \mathcal{S}_{M_p,h}(\mathbb{Z})$.

Here, the above-mentioned spaces are related to the Gelfand–Shilov space of type S in [5]. The Fourier transform on such spaces is well-defined, i.e.,

$$\widehat{\varphi}(x) = \sum_{n\in\mathbb{Z}} \varphi(n) e^{-inx} \quad \text{for } \varphi \in \mathcal{S}_*(\mathbb{Z}).$$

Here, $*$ stands for either $\{M_p\}$ or (M_p). In fact, we obtain by a straightforward computation that the Fourier transform becomes an isomorphism between these spaces.

PROPOSITION 2.1. *The* Fourier *transformation is a linear isomorphism of $\mathcal{S}_*(\mathbb{T})$ onto $\mathcal{S}_*(\mathbb{Z})$.*

The above Proposition 2.1 implies there exists a natural isomorphism between $\mathcal{S}'_*(\mathbb{T})$ and $\mathcal{S}'_*(\mathbb{Z})$ via Fourier transform, which are the strong dual spaces, respectively.

Let χ_n be the element of $\mathcal{S}_*(\mathbb{Z})$ satisfying $\chi_n(m) = \delta_{m,n}$ for each $m \in \mathbb{Z}$, where $\delta_{m,n}$ is the Kronecker delta. Then any function $\varphi \in \mathcal{S}_*(\mathbb{Z})$ may be expanded in terms of $\{\chi_n\}_{n\in\mathbb{Z}}$ as $\varphi = \sum_{n\in\mathbb{Z}} \varphi(n)\chi_n$ in $\mathcal{S}_*(\mathbb{Z})$.

Let u be an element of $\mathcal{S}'_*(\mathbb{Z})$ and put $u(n) = \langle u, \chi_n\rangle$ for any $n \in \mathbb{Z}$. Then it follows from the continuity of u that

$$\langle u, \varphi\rangle = \sum_{n\in\mathbb{Z}} u(n)\varphi(n), \quad \varphi \in \mathcal{S}_*(\mathbb{Z}).$$

Thus an element $u \in \mathcal{S}'_*(\mathbb{Z})$ is uniquely determined by and is identified with the function on $\mathbb{Z}$, defined by $n \mapsto u(n)$. Therefore, as a function on $\mathbb{Z}$ an element $u \in \mathcal{S}'_*(\mathbb{Z})$ is characterized as follows. We omit the proof.

THEOREM 2.2. *A complex-valued function u on $\mathbb{Z}$ belongs to the space $\mathcal{S}'_{\{M_p\}}(\mathbb{Z})$ if and only if u is $\{M\}$-tempered, i.e., there exists a constant $C > 0$ such that for every $\varepsilon > 0$ $|u(n)| \le Ce^{M(\varepsilon|n|)}$, $n \in \mathbb{Z}$, or if and only if u is of the form $u = \sum_{n\in\mathbb{Z}} u(n)T_n$ in the weak* topology on $\mathcal{S}'_{\{M_p\}}(\mathbb{Z})$, where T_n is the element in $\mathcal{S}'_{\{M_p\}}(\mathbb{Z})$ defined by $\langle T_n, \varphi\rangle = \varphi(n)$ for any $n \in \mathbb{Z}$ and $\varphi \in \mathcal{S}_{\{M_p\}}(\mathbb{Z})$.*

REMARK. Because of the above theorem we call elements of $\mathcal{S}'_{\{M_p\}}(\mathbb{Z})$ *$\{M\}$-tempered ultradistributions* on $\mathbb{Z}$. In fact, Theorem 2.2 is valid for the space $\mathcal{S}'_{(M_p)}(\mathbb{Z})$.

The Fourier transform $\widehat{u}$ of $u \in \mathcal{S}'_*(\mathbb{T})$ is defined by

$$\langle \widehat{u}, \widehat{\varphi}\rangle = 2\pi\langle u, \varphi\rangle \quad \text{for all } \varphi \in \mathcal{S}_*(\mathbb{T}).$$

In fact, $\mathcal{S}_*(\mathbb{T})$ is a closed subspace of the space of ultradifferentiable functions φ on $[0, 2\pi]$ satisfying $\varphi^{(k)}(0) = \varphi^{(k)}(2\pi)$, $k \in \mathbb{N}_0$ (See [6] for the definitions). Hence, $S_*(\mathbb{T})$ is separable and nuclear. Therefore, due to the linear unitary topological isomorphism of the Fourier transform, $S_*(\mathbb{T})$ and $\mathcal{S}_*(\mathbb{Z})$ are perfect spaces.

We note that the strong topology of its dual space is identical to its weak* topology and the Fourier transform of $u \in \mathcal{S}'_*(\mathbb{T})$ is the element in $\mathcal{S}'_*(\mathbb{Z})$. Therefore, we have the following isomorphism of $\mathcal{S}'_*(\mathbb{T})$ onto $\mathcal{S}'_*(\mathbb{Z})$ without proof.

THEOREM 2.3. *The* Fourier *transform is a topological isomorphism of $\mathcal{S}'_*(\mathbb{T})$ onto $\mathcal{S}'_*(\mathbb{Z})$ with respect to the weak* topology. Moreover, the following hold.*

(i) *A function T on $\mathbb{Z}$ is the* Fourier *transform $\widehat{u}$ of $u \in \mathcal{S}'_*(\mathbb{T})$ if and only if T is $*$-tempered.*

(ii) *Every $*$-tempered ultradistribution u can be expressed as* Fourier *series $u = \sum_{n\in\mathbb{Z}} \widehat{u}(-n)e^{int}$ in the weak* topology of $\mathcal{S}'_*(\mathbb{T})$.*

REMARK. In fact, using similar arguments we can obtain parallel results in the multi-dimensional case.

3. Bochner–Schwartz type theorem

We observe, as in [10], that $G_\lambda = \varprojlim_\nu (G^\nu_\lambda, h^\nu_\lambda)$ and the group G can be presented as

$$G = \varinjlim_\lambda \varprojlim_\nu (G^\nu_\lambda, h^\nu_\lambda) = \varprojlim_\nu \varinjlim_\lambda (G^\nu_\lambda, h^\nu_\lambda),$$

where indices λ and ν run through some directed sets for which G^ν_λ is defined.

The group Γ of characters on a group G is in turn obtained from elementary groups Γ^λ_ν by a consecutive direct and inverse passage to the limit executed in an arbitrary order,

$$\Gamma = \varinjlim_\nu \varprojlim_\lambda \Gamma^\lambda_\nu = \varprojlim_\lambda \varinjlim_\nu \Gamma^\lambda_\nu.$$

We shall now consider functions on groups. Any function f defined on G^ν_λ can be lifted to the group G in a natural way. In fact, we first write

$$\widetilde{f}(h^\nu_\lambda g) = f(g) \quad \text{for } g \in G^\nu_\lambda,$$

which as a function defined on an open subgroup G_λ of the group G, is extended to the whole group putting $f \equiv 0$ on $G \backslash G_\lambda$. The obtained function is invariant under translations from T^ν, where $G_\lambda / T^\nu = G^\nu_\lambda$.

A function φ on G being constant on the cosets of T^ν determines the function φ^ν_λ on the elementary group by

$$\varphi^\nu_\lambda(g) = \varphi(h^\nu_\lambda g) \quad \text{for } g \in G^\nu_\lambda.$$

If, in addition, the support of φ is contained in G_λ, then $\widetilde{\varphi^\nu_\lambda} = \varphi$. The Haar measure on G determines an invariant measure on G^ν_λ, which we denote by $(dg)^\nu_\lambda$. Thus for any integrable function f on G^ν_λ we have

$$\int_G \tilde{f}\, dg = \int_{G^\nu_\lambda} f (dg)^\nu_\lambda,$$

and if φ belongs to $L^1(G)$, then

$$\int_G \varphi\, dg = \int_{G_\lambda^\nu} \varphi_\lambda^\nu (dg)_\lambda^\nu \quad \text{only if } \operatorname{supp}\varphi \subset G_\lambda.$$

DEFINITION 3.1. If $f \in L^1(G)$, then a function $\widehat{f}$ on Γ defined by the formula

$$\widehat{f}(\gamma) = \int_G \langle g, \gamma\rangle f(g)\, dg$$

is called the *Fourier transform* of f.

Now, to introduce the Gelfand–Bruhat space of type S on an arbitrary locally compact abelian group, let G_λ^ν be an elementary group, which is identical to $\mathbb{R}^k \oplus \mathbb{T}^m \oplus \mathbb{Z}^\ell \oplus \mathbb{F}$. Then we define the *Gelfand–Bruhat space* $\mathcal{S}_{\{M_p\}}(G_\lambda^\nu)$ ($\mathcal{S}_{(M_p)}(G_\lambda^\nu)$, respectively) of type S is the set of all infinitely differentiable functions φ, defined on G_λ^ν, satisfying for some h, $h' > 0$ (for every h, $h' > 0$, respectively)

$$\sup_{G_\lambda^\nu} |\varphi(x,y,r,s)|\, e^{M(h(|x|+|r|))} < \infty$$
$$\text{and} \quad \sup_{\Gamma_\nu^\lambda} |\widehat{\varphi}(x,r,y,s)|\, e^{M(h'(|x|+|r|))} < \infty,$$

where (x,y,r,s) and (x,r,y,s) run over $G_\lambda^\nu = \mathbb{R}^k \oplus \mathbb{T}^m \oplus \mathbb{Z}^\ell \oplus \mathbb{F}$ and $\widehat{G_\lambda^\nu} = \Gamma_\nu^\lambda = \mathbb{R}^k \oplus \mathbb{Z}^m \oplus \mathbb{T}^\ell \oplus \mathbb{F}$, respectively, and $M(t)$ is the associated function of M_p. Then the Fourier transform is a linear continuous isomorphism of $\mathcal{S}_*(G_\lambda^\nu)$ onto $\mathcal{S}_*(\Gamma_\nu^\lambda)$. Moreover, we can extend $\mathcal{S}_*(G_\lambda^\nu)$ to $\mathcal{S}_*(G_\lambda)$ by the natural way as in [10]. The topology in $\mathcal{S}_*(G)$ is then defined as the strongest topology for which the injections $\mathcal{S}_*(G_\lambda) \to \mathcal{S}_*(G)$ are continuous. Moreover, the Fourier transform is a linear continuous isomorphism from $\mathcal{S}_*(G)$ onto $\mathcal{S}_*(\Gamma)$ by the open mapping theorem, where G is an arbitrary locally compact abelian group and Γ is the group of characters on G. Also, we denote by $\mathcal{S}'_*(G)$ the strong dual space of $\mathcal{S}_*(G)$ with the weak* topology. In particular, in the case of $M_p = p!$ we can define the Sato-Bruhat space $\mathcal{F}(G) = \mathcal{S}_{\{p!\}}(G)$ and call elements of the strong dual space $\mathcal{F}'(G)$ of $\mathcal{F}(G)$ *Fourier hyperfunctions* on G. In fact, for convenience we write as $\mathcal{S}'_{\{M_p\}}(\mathbb{R}^k)$ instead of the Gelfand–Shilov space $\big(\mathcal{S}_{\{M_p\}}^{\{M_p\}}\big)'(\mathbb{R}^k)$.

Note that $\mathcal{S}_*(G)$ is completely determined by $\mathcal{S}_*(G_\lambda^\nu)$ and is a separable, nuclear, and Fréchet–Schwartz space.

We shall say that a $*$-tempered ultradistribution $u \in \mathcal{S}'_*(G)$ on a group G is *positive definite* if

$$\langle u, \varphi * \varphi^* \rangle = \langle u, \int_G \varphi(g+h)\overline{\varphi(h)}\, dh \rangle \geq 0, \quad \varphi \in \mathcal{S}_*(G).$$

We are now in a position to state and prove the Bochner–Schwartz theorem for $*$-tempered ultradistributions and Fourier hyperfunctions on an arbitrary locally compact abelian group.

THEOREM 3.2. *Assume that there exist a positive integer k such that $\liminf_{p\to\infty}(m_{kp}/m_p)^2 > k$, where $m_p = M_p/M_{p-1}$, $p = 1, 2, \ldots$. If u is a positive definite ultradistribution in $S'_*(G)$ on an arbitrary locally compact abelian group G, then there exists a positive $*$-tempered measure μ on Γ such that*

$$u(\varphi) = \int_\Gamma \widehat{\varphi}(\gamma)\, d\mu(\gamma) \quad \text{for all } \varphi \in \mathcal{S}_*(G).$$

The converse is also true.

Proof. We first consider the case of the elementary group $G = \mathbb{R}^k \oplus \mathbb{T}^m \oplus \mathbb{Z}^\ell \oplus \mathbb{F}$ and prove only the case $* = \{M_p\}$.

Let u be a positive definite ultradistribution on $\mathcal{S}_{\{M_p\}}(G)$. Now, we first show that

$$u(\varphi) = \int_\Gamma \widehat{\varphi}(\gamma)\, d\mu(\gamma) \quad \text{for all } \varphi \in \mathcal{S}_{\{M_p\}}(G),$$

where μ is a positive measure on $\Gamma = \mathbb{R}^k \oplus \mathbb{Z}^m \oplus \mathbb{T}^\ell \oplus \mathbb{F}$ satisfying for every $\varepsilon > 0$

$$\int_\Gamma e^{-M(\varepsilon(|x|+|r|))}\, d\mu < \infty, \quad x \in \mathbb{R}^k,\ r \in \mathbb{Z}^m,$$

where $M(t)$ is the associated function of M_p. To begin with we note that $\mathcal{S}_{\{M_p\}}(\mathbb{R}^k) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{T}^m) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{Z}^\ell) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{F})$ is dense in $\mathcal{S}_{\{M_p\}}(\mathbb{R}^k \oplus \mathbb{T}^m \oplus \mathbb{Z}^\ell \oplus \mathbb{F})$ and the identity map is continuous. Thus, $u \in \mathcal{S}'_{\{M_p\}}(G)$ is completely determined by the restriction of u to $\mathcal{S}_{\{M_p\}}(\mathbb{R}^k) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{T}^m) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{Z}^\ell) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{F})$. Moreover, if $u \in \mathcal{S}'_{\{M_p\}}(\mathbb{T}^m)$ then u can be identified with a periodic $\{M\}$-tempered ultradistribution.

Let v be the restriction of u to $\mathcal{S}_{\{M_p\}}(\mathbb{R}^k) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{T}^m) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{Z}^\ell) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{F})$, Note that, in fact, $\mathcal{S}(\mathbb{F}) = \mathcal{S}_{\{M_p\}}(\mathbb{F}) = \mathcal{F}(\mathbb{F})$ and moreover, the positive definiteness of u implies v is positive definite $\mathcal{S}_{\{M_p\}}(\mathbb{R}^k) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{T}^m) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{Z}^\ell) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{F})$. It follows from the Bochner–Schwartz theorem for $\mathcal{S}'_{\{M_p\}}(\mathbb{R}^k)$ and $\mathcal{S}'_{\{M_p\}}(\mathbb{F})$, the Herglotz theorem for $\mathcal{S}'_{\{M_p\}}(\mathbb{T}^m)$, and the Schwartz–Bruhat theorem that we have the integral representation of v as follows:

$$v(\varphi_1 \otimes \varphi_2 \otimes \varphi_3 \otimes \varphi_4) = \int_\Gamma \widehat{\varphi}_1 \, \widehat{\varphi}_2 \, \widehat{\varphi}_3 \, \widehat{\varphi}_4 \, d\mu(\gamma).$$

Therefore, from Theorem 2.3 and [3] we have the $\{M\}$-temperedness of μ on $\mathcal{S}_{\{M_p\}}(\mathbb{R}^k) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{Z}^m) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{T}^\ell) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{F})$.

Now, for any $\varphi \in \mathcal{S}_{\{M_p\}}(G)$ we can choose a sequence $\varphi_j \in \mathcal{S}_{\{M_p\}}(\mathbb{R}^k) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{T}^m) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{Z}^\ell) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{F})$ such that $\varphi_j \to \varphi$ in $\mathcal{S}_{\{M_p\}}(G)$ as $j \to \infty$. Since the Fourier transform is an isomorphism from $\mathcal{S}_{\{M_p\}}(G)$ onto $\mathcal{S}_{\{M_p\}}(\Gamma)$, we have

$$u(\varphi) = \lim_{j\to\infty} v(\varphi_j) = \lim_{j\to\infty} \int_\Gamma \widehat{\varphi}_j \, d\mu = \int_\Gamma \widehat{\varphi} \, d\mu.$$

In fact, the existence of a positive measure μ on $\mathcal{S}_{\{M_p\}}(\Gamma)$ follows from the density of $\mathcal{S}_{\{M_p\}}(\mathbb{R}^k) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{Z}^m) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{T}^\ell) \otimes \mathcal{S}_{\{M_p\}}(\mathbb{F})$ in $\mathcal{S}_{\{M_p\}}(\Gamma)$. Of course, μ is independent of a sequence $\{\varphi_j\}$.

For the general case of an arbitrary locally compact abelian group G, we already have the family of positive measures $(d\mu)_\nu^\lambda$ on the group Γ_ν^λ of characters of elementary groups G_λ^ν, defined by

$$\int_{\Gamma_\nu^\lambda} \widehat{\varphi} \, (d\mu)_\nu^\lambda = \int_\Gamma \widehat{\widetilde{\varphi}} \, d\mu \quad \text{for } \varphi \in \mathcal{S}_{\{M_p\}}(\Gamma),$$

where Γ is the group of characters on G and $\widetilde{\varphi}$ is the extension of φ. Moreover, we have the corresponding set of positive definite ultradistributions, given by

$$u(\varphi) = \int \widehat{\varphi}(\gamma) \, d\mu(\gamma), \quad \varphi \in \mathcal{S}_{\{M_p\}}(G).$$

In virtue of the theorems proved for elementary groups, $(d\mu)_\nu^\lambda$ can be continuously extended to a $\{M\}$-tempered measure μ on Γ, which means the extension of measure $(d\mu)_\nu^\lambda$ to $\mathcal{S}_{\{M_p\}}(\Gamma)$. □

In fact, in the case of $\{M_p\} = \{p!\}$ in the above theorem we can also obtain the following Bochner–Schwartz theorem for Fourier hyperfunctions $\mathcal{F}'(G)$ on an arbitrary locally compact abelian group G.

COROLLARY 3.3. *If u is a positive definite* Fourier *hyperfunction on an arbitrary locally compact abelian group G then there exists a positive $\{p!\}$-tempered measure μ on Γ such that*

$$u(\varphi) = \int_\Gamma \widehat{\varphi}(\gamma)\, d\mu(\gamma) \quad \textit{for all } \varphi \in \mathcal{F}(G).$$

The converse is also true.

REMARK. (i) A Fourier hyperfunction u is said to be *periodic with period* p if $u(x+p) = u(x)$. Let $u = \{u_n\}_{n\in\mathbb{Z}}$ where u_n is an analytic functional on the interval $[np, (n+2)p]$ such that $u = u_n$ in $np < x < (n+2)p$. Then $u_n \in \mathcal{F}'(\mathbb{T})$ gives a complete characterization of u.

(ii) $\mathcal{S}'(\mathbb{Z}) \subsetneqq \mathcal{S}'_{\{M_p\}}(\mathbb{Z}) \subsetneqq \mathcal{S}'_{(M_p)}(\mathbb{Z}) \subsetneqq \mathcal{F}'(\mathbb{Z})$.

ACKNOWLEDGMENTS. It is a pleasure to thank Professors D. Kim and S.-Y. Chung for many fruitful conversations. Also, the authors would like to express our gratitude to the referee who has corrected many misprints and clarified places in the paper.

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