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PRECONDITIONING C^1-QUADRATIC SPLINE COLLOCATION METHOD OF ELLIPTIC EQUATIONS BY FINITE DIFFERENCE METHOD

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Abstract. We discuss a finite difference preconditioner for the C^1 Lagrange quadratic spline collocation method for a uniformly elliptic operator with homogeneous Dirichlet boundary conditions. Using the generalized field of values argument, we analyzed eigenvalues of the matrix preconditioned by the matrix corresponding to a finite difference operator with zero boundary condition.

1. Introduction

Let A be a uniformly elliptic invertible operator defined on the unit square $\Omega := I \times I$, where $I = [0, 1]$, of the form

$$Au := -\Delta u + a_0(x, y)u, \tag{1.1}$$

with homogeneous Dirichlet boundary condition, where the coefficient $a_0(x, y)$ is a nonnegative bounded continuous function. The orthogonal collocation method for differential equations was investigated by many authors (for example, see [1], [3], [11], and [12]) for a problem

$$Au = f \tag{1.2}$$

with various boundary conditions. The iterative line spline collocation method was reported in [5] where spectral radius of the Jacobi iteration matrix are studied and a spectrum for Hermite cubic spline collocation

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method is analyzed in [15]. For the orthogonal spline collocation method, fast direct solvers were developed in [1]. Also fast algorithms were reported for high-order spline collocation systems in [14]. By contrast with recent developments of fast direct solver, a preconditioning technique related to the usual finite difference method for a quadratic spline collocation method is considered here. The spline collocation method has a property such that the condition number of the matrix $\hat{A}_{N^2}$ increases as a power of $1/h(h = 1/N)$. Thus it is necessary to bound for such condition numbers which are independent of the size of a preconditioned matrix, for the successful applications of the well known iterative methods such as damped Jacobi iterative method, GMRES(see [13]), Conjugate Gradient method and etc. In this sense, one can consider a preconditioning technique using finite difference method for C^1-Lagrange quadratic polynomial spline collocation method, which is to find such a spline solution $u(x,y)$ which satisfies

$$Au(\xi_i, \xi_j) = f(\xi_i, \xi_j)$$

and boundary conditions, where the collocation points $\{(\xi_i, \xi_j)\}_{i,j=1}^N$ are chosen as the Legendre-Gauss[=:LG] points (see [3]). One may use Matlab(see [2]) for the construction of C^1-quadratic spline to solve (1.2) easily. Let $\{A_{N^2}\}$ be the collocation discretization operator of A based on the C^1-Lagrange quadratic polynomial spline space and the local LG points and $\hat{A}_{N^2}$ be its matrix representation. For the C^1-cubic spline collocation method, the uniform bounds of the condition numbers was investigated in [8] or [9], respectively. The multigrid and multilevel methods are studied for quadratic spline collocation method in [4]. We note that the exponential decay of C^1-polynomial spline is important for the preconditioning theory by finite difference or element method(see [7]).

In the paper, we choose a preconditioning operator L of the form

$$Lv := -\Delta v \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

Let L_{N^2} be the finite difference discretization of L using the local LG points to define the local rectangles and $\hat{L}_{N^2}$ be its matrix representation of L_{N^2}.

The main result is to get a uniform bound for the eigenvalues of the preconditioned matrix $\hat{L}_{N^2}^{-1}\hat{A}_{N^2}$ for the Helmholtz operator A such that $Au = -\Delta u$. Such an estimate is important for the successful application of the damped Jacobi iteration. In Section 2, we collect some preliminary ideas, notations, etc. In Section 3, the $2D$ preconditioning problems are analyzed .

2. Gauss points, Spaces and Basis

Let $h = \frac{1}{N}$, where N is a nonzero positive integer. The knots are given by the points $x_i := ih (i = 0, 1, \ldots, N)$ and the i^{th}- subinterval is denoted by $I_i := [x_{i-1}, x_i] (i = 1, 2, \cdots, N)$. Let $\{\xi_i\}_{i=1}^N$ be the set of local LG points such that $\xi_i = x_{i-1} + \frac{h}{2}$. With $\xi_0 = 0$ and $\xi_{N+1} = 1$, define $S_{h,1}$ as the space of continuous piecewise linear functions on the unit interval whose restriction on each subinterval $[\xi_i, \xi_{i+1}], (i = 0, 1, \cdots, N)$ is linear satisfying the zero boundary conditions. The basis functions for $S_{h,1}$ are given by the usual hat functions $\{\phi_k\}_{k=1}^N$ satisfying $\phi_k(\xi_l) = \delta_{k,l}, \quad l = 0, 1, \cdots, N+1$. The two dimensional space $S_{h^2,1}$ is defined by the tensor product of two one-dimensional spaces $S_{h,1}$. The basis functions $\{\Phi_\mu(x,y), \mu = 1, 2, \cdots, N^2\}$ of $S_{h^2,1}$ are given by $\Phi_\mu(x,y) := \phi_k(x)\phi_l(y), \; \mu = k + N(l-1)$. Let $S_{h,2}$ be the space of C^1 functions whose restriction on each I_i is quadratic polynomial satisfying zero boundary conditions. In this paper, we use the C^1 Lagrange quadratic spline basis of $S_{h,2}$ which consists of the functions $\psi_i \in S_{h,2}, \; i = 1, 2, \cdots, N$, satisfying $\psi_i(\xi_k) = \delta_{k,j}, \; 0 \le k \le N+1$.

The two dimensional spaces $S_{h^2,2}$ is given as the tensor product of the appropriate one dimensional spaces, which has the basis functions $\{\Psi_\mu\}_{\mu=1}^{N^2}$, $\Psi_\mu(x,y) := \psi_i(x)\psi_j(y), \; \mu = i + (j-1)N$.

We define the $1D$ quadratic spline interpolant $I_N u$ of u as

$$I_N u(t) = \sum_{i=1}^N u_i \psi_i(t) \in S_{h,2} \text{ if } u(t) = \sum_{i=1}^N u_i \phi_i(t) \in S_{h,1}.$$

In a similar way, we can define a $2D$ quadratic spline interpolant I_{N^2}.

Define, for continuous functions u and v,

$$\langle u, v\rangle_{N^2} = h^2 \sum_{\mu=1}^{N^2} u(\xi_i, \xi_j)\ v(\xi_i, \xi_j).$$

Note that A_{N^2} is the quadratic collocation operators mapping $S_{h^2,2}$ to itself corresponding to (1.1) with zero boundary condition. The quadratic spline collocation discretization A_{N^2} in the space $S_{h^2,2}$ with the basis $\{\Psi_\mu\}_{\mu=1}^{N^2}$ is determined by the following bilinear form

$$a_{N^2}(u,v) := \langle A_{N^2}u, v\rangle_{N^2}.$$

For any $u, v \in S_{h^2,2}$, we may have

$$u(x,y) = \sum_{\mu=1}^{N^2} u_\mu \Psi_\mu(x,y) \quad \text{and} \quad v(x,y) = \sum_{\mu=1}^{N^2} v_\mu \Psi_\mu(x,y).$$

Let $U = (u_1, \cdots, u_{N^2})^t$ and $V = (v_1, \cdots, v_{N^2})^t$. Then one can easily check that $a_{N^2}(u,v) = h^2 V^* \hat{A}_{N^2} U$, where $\hat{A}_{N^2}$ is the corresponding matrix representation of A_{N^2}. We will denote $a_N \sim b_N$ if there are two positive constants α and β, independent of N, such that for all N, $0 < \alpha a_N < b_N < \beta a_N$. Let $\{u_i\}_{i=1}^N$ be such that $u_i := u(\xi_i)$, $i = 1, 2, \cdots, N$, where ξ_i is the local LG points in I. Then the one dimensional second order central finite difference operator corresponding to $-u''$ is given by

$$[L_N u]_k := \frac{2}{h_k + h_{k-1}} \Big\{ -\frac{u_{k+1}}{h_k} + \Big(\frac{1}{h_k} + \frac{1}{h_{k-1}}\Big) u_k - \frac{u_{k-1}}{h_{k-1}} \Big\},$$

where $h_k := \xi_{k+1} - \xi_k, (k = 0, 1, \cdots, N)$.

3. Eigenvalue results

In this section we will compare the finite difference scheme defined in the space $S_{h^2,1}$ corresponding to

$$Lu := -(u_{xx} + u_{yy}) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

with the usual Sobolev H^1 norm of u.

Let $\{u_\mu\}_{\mu=1}^{N^2}$ be such that

$$u_{k,l} = u_\mu := u(P_\mu), \quad P_\mu = (\xi_k, \xi_l), \quad \mu = k + (l-1)N,$$

where P_μ is the local LG points in Ω.

The finite difference operator L_{N^2} corresponding to L on $S_{h^2,1}$ can be written as

$$[L_{N^2}u]_{k,l} := \frac{2}{h_k + h_{k-1}}\Big\{ - \frac{u_{k+1,l}}{h_k} + \Big(\frac{1}{h_k} + \frac{1}{h_{k-1}}\Big) u_{k,l} - \frac{u_{k-1,l}}{h_{k-1}}\Big\} + \frac{2}{h_l + h_{l-1}}\Big\{ - \frac{u_{k,l+1}}{h_l} + \Big(\frac{1}{h_l} + \frac{1}{h_{l-1}}\Big) u_{k,l} - \frac{u_{k,l-1}}{h_{l-1}}\Big\}.$$

Define the bilinear form on $S_{h^2,1} \times S_{h^2,1}$ as

$$l_{N^2}(u,v) := h^2 \sum_{i,j=1}^{N} [L_{N^2}u]_{i,j} \bar{v}_{i,j}\,, \tag{3.1}$$

where $\bar{v}$ denotes the complex conjugate of v.

Let, for u_1 and u_2 in $\mathbf{C}$, define

$$f(u_1,u_2) := \frac{1}{h_0} u_1 \bar{v_1} + \frac{1}{h_1}(u_2 - u_1)(\bar{v_2} - \bar{v_1})$$
$$g(u_1,u_2) := \frac{4}{3h_0} u_1 \bar{v_1} + \frac{1}{h_1}(u_2 - u_1)(\bar{v_2} - \frac{4}{3}\bar{v_1}).$$

One can verify easily that

$$\begin{aligned} &\mathrm{Re}(g(u_1,u_2)) \sim f(u_1,u_2) \\ &|\mathrm{Im}(g(u_1,u_2))| \le C f(u_1,u_2), \end{aligned} \tag{3.2}$$

where C is a positive constant.

Note that the bilinear form $l_{N^2}(u,v)$ defined in (3.1) can be written as, using the changes of indices and boundary conditions,

(3.3)

$$\begin{aligned} &\ell_{N^2}(u,v) \\ &= h \sum_{j=1}^{N} [g(u_{1,j}, u_{2,j}) + \sum_{i=2}^{N-2} \frac{(u_{i+1,j} - u_{i,j})(\bar{v}_{i+1,j} - \bar{v}_{i,j})}{h_i} \\ &\quad + g(-u_{N,j}, -u_{N-1,j})] \\ &\quad + h \sum_{i=1}^{N} [g(u_{i,1}, u_{i,2}) + \sum_{j=2}^{N-2} \frac{(u_{i,j+1} - u_{i,j})(\bar{v}_{i,j+1} - \bar{v}_{i,j})}{h_j} \\ &\quad + g(-u_{i,N}, -u_{i,N-1})]. \end{aligned}$$

For the continuity, we need a simple lemma.

LEMMA 3.1. *If f is a linear function on $[a,b]$, then there are positive numbers $C_i, i=1,2$, such that*

$$C_1\int_a^b f(x)^2dx \le \frac{b-a}{2}\{f(a)^2+f(b)^2\} \le C_2\int_a^b f(x)^2dx.$$

Proof. Without loss of generality, we may assume $a=0, b=h>0, f(x)=\xi x+\eta$. It suffices to show that there are $C_i, i=1,2$ such that $C_1B\le A\le C_2B$, where

$$A=\frac{h}{2}\{\eta^2+(\xi h+\eta)^2\}=\frac{\xi^2}{2}h^3+\xi\eta h^2+\eta^2 h$$
$$B=\frac{\xi^2}{3}h^3+\xi\eta h^2+\eta^2 h.$$

By a simple comparision, the first inequality is obvious with $C_1=1$. For the second inequality, using the inequality $ab\le\frac{1}{2\epsilon}a^2+\frac{\epsilon}{2}b^2$, we have

$$\begin{aligned}C_2B-A&=(\frac{C_2}{3}-\frac{1}{2})\xi^2h^3+(C_2-1)\xi\eta h^2+(C_2-1)\eta^2h\\&\ge(\frac{C_2}{3}-\frac{1}{2}-\frac{C_2-1}{2\epsilon})\xi^2h^3+(C_2-1-\frac{(C_2-1)\epsilon}{2})\eta^2h.\end{aligned}$$

We need to find $C_2>0$ with some positive $\epsilon>0$ with

$$C_2(\frac{1}{3}-\frac{1}{2\epsilon})-\frac{1}{2}+\frac{1}{2\epsilon}\ge 0 \text{ and } C_2(1-\frac{\epsilon}{2})-1+\frac{\epsilon}{2}\ge 0.$$

Now it is easy to find $C_2>0$ with $\frac{3}{2}<\epsilon<2$. □

PROPOSITION 3.2. *For $u,v\in S_{h^2,1}$, there is a positive constant C_3, independent of h, such that*

$$|\ell_{N^2}(u,v)|\le C_3||u||_1||v||_1. \tag{3.4}$$

Proof. We have, from (3.3),

$$
\begin{aligned}
&| \ell_{N^2}(u,v)| \\
&\le h\sum_{j=1}^{N}[|g(u_{1,j},u_{2,j})| + \sum_{i=2}^{N-2}\frac{|u_{i+1,j}-u_{i,j}||\bar v_{i+1,j}-\bar v_{i,j}|}{h_i} \\
&\quad + |g(-u_{N,j},-u_{N-1,j})|] \\
&\quad + h\sum_{i=1}^{N}[|g(u_{i,1},u_{i,2})| + \sum_{j=2}^{N-2}\frac{|u_{i,j+1}-u_{i,j}||\bar v_{i,j+1}-\bar v_{i,j}|}{h_j} \\
&\quad + |g(-u_{i,N},-u_{i,N-1})|].
\end{aligned}
$$

By (3.2), we have $|g(u_1,u_2)| \le Cf(u_1,u_2)$, where C is an absolute positive constant. Therefore, using the fact $u \in S_{h,1}$ and the boundary condition, we have

$$
\begin{aligned}
&|\ell_{N^2}(u,v)| \\
&\le Ch\sum_{j=1}^{N}\Big\{\Big(\sum_{i=0}^{N}\frac{1}{h_i}|u_{i+1,j}-u_{i,j}|^2\Big)^{\frac12}\Big(\sum_{i=0}^{N}\frac{1}{h_i}|\bar v_{i+1,j}-\bar v_{i,j}|^2\Big)^{\frac12}\Big\} \\
&\quad + Ch\sum_{i=1}^{N}\Big\{\Big(\sum_{j=0}^{N}\frac{1}{h_j}|u_{i,j+1}-u_{i,j}|^2\Big)^{\frac12}\Big(\sum_{j=0}^{N}\frac{1}{h_j}|\bar v_{i,j+1}-\bar v_{i,j}|^2\Big)^{\frac12}\Big\} \\
&= Ch\sum_{j=1}^{N}\Big(\sum_{i=0}^{N}\int_{\xi_i}^{\xi_{i+1}}|u_x(\cdot,\xi_j)|^2dx\Big)^{\frac12}\Big(\sum_{i=0}^{N}\int_{\xi_i}^{\xi_{i+1}}|v_x(\cdot,\xi_j)|^2dx\Big)^{\frac12} \\
&\quad + Ch\sum_{i=1}^{N}\Big(\sum_{j=0}^{N}\int_{\xi_j}^{\xi_{j+1}}|u_y(\xi_i,\cdot)|^2dy\Big)^{\frac12}\Big(\sum_{j=0}^{N}\int_{\xi_j}^{\xi_{j+1}}|v_y(\xi_i,\cdot)|^2dy\Big)^{\frac12} \\
&= Ch\sum_{j=1}^{N}\Big(\int_0^1|u_x(\cdot,\xi_j)|^2dx\Big)^{\frac12}\Big(\int_0^1|v_x(\cdot,\xi_j)|^2dx\Big)^{\frac12} \\
&\quad + Ch\sum_{i=1}^{N}\Big(\int_0^1|u_y(\xi_i,\cdot)|^2dy\Big)^{\frac12}\Big(\int_0^1|v_y(\xi_i,\cdot)|^2dy\Big)^{\frac12}.
\end{aligned}
$$

If we use Cauchy Schwarz inequality, the boundary condition, and Lemma

3.1, we have

$$
\begin{aligned}
|\ell_{N^2}(u,v)| \le\, & dh\Big(\sum_{j=1}^{N}\int_0^1 |u_x(\cdot,\xi_j)|^2 dx\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{N}\int_0^1 |v_x(\cdot,\xi_j)|^2 dx\Big)^{\frac{1}{2}} \\
& + dh\Big(\sum_{i=1}^{N}\int_0^1 |u_y(\xi_i,\cdot)|^2 dy\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{N}\int_0^1 |v_y(\xi_i,\cdot)|^2 dy\Big)^{\frac{1}{2}} \\
\le\, & d\Big(\int_0^1 \sum_{j=0}^{N} \frac{h}{2}[|u_x(\cdot,\xi_j)|^2 + |u_x(\cdot,\xi_{j+1})|^2]dx\Big)^{\frac{1}{2}} \\
& \cdot\Big(\int_0^1 \sum_{j=0}^{N} \frac{h}{2}[|v_x(\cdot,\xi_j)|^2 + |v_x(\cdot,\xi_{j+1})|^2]dx\Big)^{\frac{1}{2}} \\
& + d\Big(\int_0^1 \sum_{i=0}^{N} \frac{h}{2}[|u_y(\xi_i,\cdot)|^2 + |u_y(\xi_{i+1},\cdot)|^2]dy\Big)^{\frac{1}{2}} \\
& \cdot\Big(\int_0^1 \sum_{i=0}^{N} \frac{h}{2}[|v_y(\xi_i,\cdot)|^2 + |v_y(\xi_{i+1},\cdot)|^2]dy\Big)^{\frac{1}{2}} \\
\le\, & C\Big(\int_\Omega u_x^2\Big)^{\frac{1}{2}}\Big(\int_\Omega v_x^2\Big)^{\frac{1}{2}} + C\Big(\int_\Omega u_y^2\Big)^{\frac{1}{2}}\Big(\int_\Omega v_y^2\Big)^{\frac{1}{2}} \\
\le\, & C\Big\{||u_x||^2 + ||u_y||^2\Big\}^{\frac{1}{2}}\Big\{||v_x||^2 + ||v_y||^2\Big\}^{\frac{1}{2}} \\
\le\, & C_3||u||_1||v||_1,
\end{aligned}
$$

where C_3 is a constant. □

PROPOSITION 3.3. *For $u \in S_{h^2,1}$, there is a positive constant C_4, independent of h, such that*

$$
|\mathrm{Re}(\ell_{N^2}(u,u))| \ge C_4||u||_1^2. \tag{3.5}
$$

Proof. We have, from (3.3),

$$
\begin{aligned}
& |\mathrm{Re}(\ \ell_{N^2}(u,u))| \\
=& h\sum_{j=1}^{N}\Big[|g(u_{1,j},u_{2,j})| + \sum_{i=2}^{N-2}\frac{|u_{i+1,j}-u_{i,j}|^2}{h_i} + |g(-u_{N,j},-u_{N-1,j})|\Big]
\end{aligned}
$$

$$+h\sum_{i=1}^{N}\Big[|g(u_{i,1},u_{i,2})|+\sum_{j=2}^{N-2}\frac{|u_{i,j+1}-u_{i,j}|^2}{h_j}+|g(-u_{i,N},-u_{i,N-1})|\Big].$$

By (3.2), we have $\mathrm{Re}g(u_1,u_2)\geq cf(u_1,u_2)$, where c is an absolute positive constant. Therefore, using the fact $u_x(\cdot,\xi_j)$ and $u_y(\xi_i,\cdot)$ are piecewise constants, and the boundary condition and Lemma 3.1, we have

$$\begin{aligned}
&|\mathrm{Re}(\ell_{N^2}(u,u))|\\
&\geq ch\sum_{j=1}^{N}\sum_{i=0}^{N}\frac{1}{h_i}|u_{i+1,j}-u_{i,j}|^2+ch\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{1}{h_j}|u_{i,j+1}-u_{i,j}|^2\\
&=ch\sum_{j=1}^{N}\sum_{i=0}^{N}\int_{\xi_i}^{\xi_{i+1}}|u_x(\cdot,\xi_j)|^2dx+ch\sum_{i=1}^{N}\sum_{j=0}^{N}\int_{\xi_j}^{\xi_{j+1}}|u_y(\xi_i,\cdot)|^2dy\\
&=ch\sum_{j=1}^{N}\int_0^1|u_x(\cdot,\xi_j)|^2dx+ch\sum_{i=1}^{N}\int_0^1|u_y(\xi_i,\cdot)|^2dy\\
&\geq c\int_0^1\sum_{j=0}^{N}\frac{h_j}{2}\Big[|u_x(\cdot,\xi_j)|^2+|u_x(\cdot,\xi_{j+1})|^2\Big]dx\\
&\quad+c\int_0^1\sum_{i=0}^{N}\frac{h_i}{2}\Big[|u_y(\xi_i,\cdot)|^2+|u_y(\xi_{i+1},\cdot)|^2\Big]dy\\
&\geq c\int_\Omega u_x^2+c\int_\Omega u_y^2=C\Big[||u_x||^2+||u_y||^2\Big]=C||\nabla u||^2,
\end{aligned}$$

where C is a constant. So, by Poincaré inequality, we have the Proposition. □

From Proposition 3.2, there exists also a positive constant C_5, independent of h, such that $|\ell_{N^2}(u,u)|\leq C_5||u||_1^2$. We can show the following lemma by using the arguments of [10, Theorem 3.2]

LEMMA 3.4. *Let $u=p+iq$, where the real functions p and q are in $S_{h^2,1}$. Then there are positive constants c and C such that*

$$c||u||_1^2\leq\langle-\Delta(I_{N^2}u),I_{N^2}u\rangle_{N^2}\leq C||u||_1^2,$$

$$||I_{N^2}u||_{N^2}\sim||u||_{L^2},$$

and

$$|\langle (I_{N^2}u)_t, (I_{N^2}v)\rangle_{N^2}| \le ||u_t||_{L^2}||v||_{L^2}.$$

Proof. It can be easily verified following the real case in [6]. □

From now on, we will show the eigenvalue analysis for the preconditioned system $L_{N^2}^{-1}A_{N^2}$.

THEOREM 3.5. *Let $\lambda_1, \lambda_2, \ldots, \lambda_{N^2}$ be the eigenvalues of $\hat{L}_{N^2}^{-1}\hat{A}_{N^2}$, then*

$$\mathrm{Re}(\lambda_k) \ge C_6 > 0,$$

and

$$|\lambda_k| \le C_7 < \infty$$

for all $k = 1, 2, \cdots, N^2$.

Proof. For a nonzero complex vector $U = (u_1, \cdots, u_{N^2})^t$, First note that $(\hat{L}_{N^2}U, U)$ is complex and $(\hat{A}_{N^2}U, U)$ is real. Furthermore we have two positive constants C_6' and C_7', independent of N, such that

$$C_6'||u||_1^2 \le a_{N^2}(I_{N^2}U, I_{N^2}U) = h^2U^*\hat{A}_{N^2}U \le C_7'||u||_1^2.$$

and

$$\frac{(\hat{A}_{N^2}U, U)}{(\hat{L}_{N^2}U, U)} = a_{N^2}(I_{N^2}u, I_{N^2}u)\frac{\mathrm{Re}(\ell_{N^2}(u,u)) - i\mathrm{Im}(\ell_{N^2}(u,u))}{|\ell_{N^2}(u,u)|^2}.$$

Thus

$$\mathrm{Re}\left(\frac{(\hat{A}_{N^2}U, U)}{(\hat{L}_{N^2}U, U)}\right) = a_{N^2}(I_{N^2}u, I_{N^2}u)\frac{\mathrm{Re}(\ell_{N^2}(u,u))}{|\ell_{N^2}(u,u)|^2} \ge \frac{C_6'C_4}{C_5^2} := C_6 \tag{3.6}$$

and

$$\left|\frac{(\hat{A}_{N^2}U, U)}{(\hat{L}_{N^2}U, U)}\right| \le \frac{a_{N^2}(I_{N^2}u, I_{N^2}u)}{\mathrm{Re}(\ell_{N^2}(u,u))} \le \frac{C_7'}{C_4} := C_7. \tag{3.7}$$

Let (λ_k, U_k) be an eigen-pair of $\hat{L}_{N^2}^{-1}\hat{A}_{N^2}$ so that

$$\hat{A}_{N^2}U_k = \lambda_k\hat{L}_{N^2}U_k$$

and

$$\lambda_k = \frac{(\hat{A}_{N^2}U_k, U_k)}{(\hat{L}_{N^2}U_k, U_k)}.$$

Then the conclusion follows from (3.6) and (3.7). □

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